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Derivative expansion for the effective action of chiral gauge fermions. The abnormal parity component

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Abstract. Explicit exact formulas are presented, for the leading order term in a strict chiral covariant derivative expansion, for the abnormal parity component of the effective action of two- and four-dimensional Dirac fermions in the presence of scalar, pseudo-scalar, vector and axial vector background fields. The formulas hold for completely general internal symmetry groups and general configurations. In particular, the scalar and pseudo-scalar fields need not be on the chiral circle.

1 Introduction

This paper is the second part of the work initiated in [1] on the explicit computation of the effective action of chiral gauge fermions, including scalar, pseudo-scalar, vector and axial vector external fields, within a strict covariant derivative expansion. Reference [1] dealt with the real part of the effective action and here the imaginary part is worked out at leading order for two- and four-dimensional fermions. The main feature of both works is that a strict covariant derivative expansion is carried out, rather than a perturbative, commutator or heat-kernel expansion, and that explicit formulas are given which hold without putting any restrictions on the external field configurations, nor are we making assumptions on the internal symmetry group. In fact, this generality helps to concentrate on the computational issues and results in an easier calculation.

The imaginary part of the effective action of chiral gauge fermions (the phase of the fermionic determinant) displays some well-known peculiarities as compared to the real part. It presents a $2\pi\mathrm{i}$ multivaluation, anomalies in the chiral symmetry and contains topological pieces. In comparison the real part only displays a scale anomaly, which however is absent in the imaginary part. These peculiarities make this piece more interesting from the theoretical point of view and has been the source of deeply original insights [2–10]. Consequently, it has extensively been studied in the literature (for reviews see e.g. [11,12].)

The presence of the chiral anomaly introduces some mathematical subtleties in the definition of the effective action at the non-perturbative level [11] since the chirally covariant renormalized current (the variation of the effective action) fails to be consistent [13]. These complications are also present in the computation of the effective action in the framework of an asymptotic expansion, such as the covariant derivative expansion to be considered here. A direct computation must necessarily break chiral invariance and becomes prohibitive if one insists on a strict derivative expansion except for particular internal symmetry groups. The reason is that in a strict covariant derivative expansion both the scalar and the pseudo-scalar fields must be treated non-perturbatively, and as a rule it not possible to treat two or more operators non-perturbatively unless they commute. For instance, in [14] such a calculation is done for two-dimensional fermions with SU(2) internal symmetry group. In that case the particular algebraic properties of su(2) allowed one to carry out the computation, but the same method cannot be extended to general groups.

An alternative method is to make a chiral rotation to fix the chiral gauge so that there is no pseudo-scalar field. Then a direct calculation becomes possible, using for instance a ζ-function approach combined with a symbols method [14]. Because the chiral gauge has been fixed (or rather, reduced to a manageable vector gauge invariance), such as result is, in some sense, manifestly chiral gauge invariant (the anomaly comes through the Wess–Zumino–Witten term generated by the chiral rotation). However, this procedure is not completely satisfactory for various reasons. The result would be given in terms of the rotated variables rather than in terms of the original external fields. In addition, it does not fully exploit the symmetries of the problem; as will be shown, within the derivative expansion the effective action depends analytically on the external fields, in a sense to be made more precise below, and this property is not explicit in terms of the rotated variables. Analyticity is a property of the effective action functional which is not shared by most functionals that are chiral invariant (modulo anomalies). Another important shortcoming of that method, as compared to the method to be presented here, is that the functional depends on three objects, the (rotated) scalar field S, the axial field A and the vector gauge covariant derivative D_V, whereas in our approach there are just two objects, an effective scalar field m and an effective vector gauge covariant derivative D which behave almost as those of a vector-like theory (i.e., a theory without pseudo-scalar nor axial vector fields). This results in a great reduction of the amount of algebra required, due to the smaller number of algebraic combinations and the fact that analyticity is preserved throughout.

The method proposed in this work is based on using a suitable notation which allows one to map certain chiral invariant objects with an analytical form (and in particular the chiral covariant effective current) to the corresponding object in an effective vector-like theory. This allows us to carry out computations in the effective vector-like theory and then map back the result into the chiral setting. This is a kind of analytical extension from the vector-like case to the full chiral case and the chiral result so obtained corresponds to the LR version of the effective action. This procedure is very convenient from the computational point of view since the vector-like case is very well understood and many methods exist to deal with it. In particular, the issue of renormalization is almost trivial since the requirement of (effective) vector gauge invariance completely fixes the form of the effective action.

For the normal parity component the mapping between chiral and vector theories is literal and applies to the effective action itself. This is exploited in [1]. In the abnormal parity sector the mapping from vector to chiral holds whenever the trace cyclic property is not involved, e.g. for the covariant effective current, but not for the effective action itself, since this would not allow for the existence of the chiral anomaly. In our formalism this is reflected in the fact that m is odd under cyclic transformations. Our strategy will then be along the lines of Schwinger's method [15], i.e., we first compute the covariant effective current and subsequently use this current to recover the effective action. This second step is done by writing down an explicit analytical functional of the type of the Wess–Zumino–Witten term which saturates the chiral symmetry breaking terms of the effective action, and adjusting the remainder, which necessarily will be chiral invariant, so that the correct current is reproduced. The calculation of the current is done from scratch by using essentially the method of symbols, but in the improved version due to Pletnev and Banin [16] which reduces the amount of algebra while preserving explicit gauge invariance throughout.

In Sect. 2 we recall the notation introduced in [1] and extend it to cover the abnormal parity case. A set of notational conventions are introduced so that the chiral case can, to a large extent, be treated as a vector theory. A further convention is introduced which allows us to carry out explicit loop momentum integrations without assuming commutativity of the operators involved. This convention is illustrated in the same section with the Wess–Zumino–Witten action which is brought into an explicit Lagrangian form preserving manifest global vector gauge invariance, for a general gauge group. In Sect. 3 the chiral covariant effective current is explicitly computed at leading order in the derivative expansion using the method of symbols for the two- and four-dimensional cases. In Sect. 4 we introduce an extended version of the gauged Wess–Zumino–Witten action which holds off the chiral circle and depends analytically on the external fields. Next we consider the general form of the possible chiral invariant remainder (which saturates the full abnormal parity effective action at leading order). This remainder is then explicitly determined from the current. In Sect. 5 several comments and extensions are given. In Sect. 5.1 we show that on the chiral circle our extended gauge Wess–Zumino–Witten term reduces to the usual one and the chiral invariant remainder vanishes. This result is extended to the case of an Abelian chiral radius and in particular to the full Abelian case. In Sect. 5.2 the effective density (the variation of the effective action with respect the scalar and pseudo-scalar fields) is explicitly computed and the anomalous continuity equation verified. Both for the current and for the density an unexpected extra symmetry is found which does not follow from Lorentz and chiral symmetries but seems to depend on the concrete properties of the effective action functional. In Sect. 5.3 the VA version of the effective action is considered and the corresponding formulas are given for the particular case of a vanishing pseudo-scalar field. In Sect. 5.4 we show that the imaginary part of the effective action vanishes when one of the matter chiral fields is a spacetime constant and there are no chiral gauge fields. Next we show how this observation, plus the assumption of analyticity, is sufficient to completely determine the effective action in two dimensions and puts restrictions in higher dimensions. In Sects. 5.5 and 5.6 we consider further properties of the extended gauged Wess–Zumino–Witten term and of the chiral invariant remainder. Finally, in Sect. 5.7 we verify a descent relation which relates the effective action in d dimensions with the vector current in $d+2$ dimensions and the Chern–Simons term in $d+1$ dimensions. In Appendix A we collect the formulas corresponding to the chiral anomaly and the various versions of the Wess–Zumino–Witten action and Appendix B contains the explicit formulas for the effective action and the effective current in two and four dimensions.

2 Notation and conventions

We will follow the notation and conventions summarized in Sect. 2 of [1]. The extensions needed to adapt these conventions to the pseudo-parity odd case are presented below. (Reference [1] deals with the pseudo-parity even component of the effective action, W^+.) Because some of these conventions are not standard, the reader is invited to consult Sect. 2 of [1] for further details.

The spacetime is Euclidean and flat and its dimension d is even. The class of Dirac operators to be considered is

$$\mathrm{D} = \not{D}_{\mathrm{R}}\, P_{\mathrm{R}} + \not{D}_{\mathrm{L}}\, P_{\mathrm{L}} + m_{\mathrm{LR}} P_{\mathrm{R}} + m_{\mathrm{RL}} P_{\mathrm{L}} \qquad (1)$$

where $P_{\mathrm{R,L}} = \frac{1}{2}(1\pm\gamma_5)$ are the projectors on the subspaces $\gamma_5 = \pm 1$. Our conventions are

$$\gamma_\mu = \gamma_\mu^\dagger, \quad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu},$$
$$\gamma_5 = \gamma_5^\dagger = \gamma_5^{-1} = \eta_d \gamma_0 \cdots \gamma_{d-1}, \quad \mathrm{tr}_{\mathrm{Dirac}}(1) = 2^{d/2}, \qquad (2)$$

where $\eta_d = \pm\mathrm{i}^{d/2}$ (a concrete choice will not be needed except in Sect. 5.7). $D_\mu^{\mathrm{R,L}} = \partial_\mu + v_\mu^{\mathrm{R,L}}$ are the chiral covariant derivatives. The external bosonic fields $v_\mu^{\mathrm{R,L}}(x)$ and $m_{\mathrm{LR}}(x)$, $m_{\mathrm{RL}}(x)$ are matrices in some generic internal space (referred to as flavor), the identity in Dirac space and multiplicative operators in x space. In order to avoid infrared divergences we will assume that the matrices m_{LR} and m_{RL} are nowhere singular. No algebraic assumptions will be made on the internal space matrices in the derivation of our results. Of course, at the end, they can be applied to particular interesting cases, such as Abelian groups, or the case of scalar fields on the so-called chiral circle, $m_{\mathrm{LR}}(x)m_{\mathrm{RL}}(x) = M^2$, M^2 being a constant c-number, for which many results exist.

In what follows, the symbol $\langle\ \rangle$ will we used as a shorthand to denote

$$\langle X\rangle_{d,B} = \frac{\eta_d (d/2)!}{(2\pi)^{d/2} d!}\int_B \mathrm{tr}(X). \qquad (3)$$

In this formula d is the space-time dimension, η_d is the normalization in γ_5, tr refers to flavor only, B is some n-dimensional integration region, and X is some differential n-form which is a matrix in flavor space. In general B will be the spacetime and X a d-form, and the subscripts B and d will be suppressed.

2.1 Specific conventions

The effective action is a functional of the external fields, defined as $W[v,m] = -\mathrm{Tr}\log(\mathrm{D})$, where some regularization plus renormalization is understood. The pseudo-parity transformation is defined as the operation of exchanging the chiral labels R and L everywhere. The effective action then decomposes naturally into a pseudo-parity even (or normal parity) component, $W^+[v,m]$, and a pseudo-parity odd (or abnormal parity) one, $W^-[v,m]$. The latter is also characterized by being purely imaginary (in Euclidean space), containing the Levi-Civita pseudotensor, having topological pieces, displaying multivaluation by integer multiples of $2\pi\mathrm{i}$, and presenting an essential anomaly under chiral transformations.

The effective action can be expanded into terms with a well-defined number of covariant derivatives (or equivalently, of Lorentz indices). For each such term T, one can consider its pseudo-parity conjugate T^*, i.e., the same expression as T after the exchange of all labels L with R. Then we will adopt the following convention (see Sect. 2 of [1] for further details):

Convention 1. In W^+, the terms T and T^* will be identified, so that under this convention T actually stands for $\frac{1}{2}(T+T^*)$. In W^- every term T is identified with $-T^*$ and thus T stands for $(1/2)(T-T^*)$.

Consider now a typical chiral invariant expression such as $\mathrm{tr}(F_{\mu\nu}^{\mathrm{R}}\hat{D}_\mu m_{\mathrm{RL}}\hat{D}_\nu m_{\mathrm{LR}})$. (As usual,

$$\hat{D}_\mu m_{\mathrm{RL}} = D_\mu^{\mathrm{R}} m_{\mathrm{RL}} - m_{\mathrm{RL}} D_\mu^{\mathrm{L}}, \quad F_{\mu\nu}^{\mathrm{R}} = [D_\mu^{\mathrm{R}}, D_\nu^{\mathrm{R}}],$$

etc.) It can be observed that each factor falls into one of the following classes, according to its chiral labels, namely RR, LL, RL and LR. For instance, $\hat{D}_\nu m_{\mathrm{LR}}$ lies in the class LR. By inserting such a factor in an expression, the chiral label is flipped from R to L as one moves from right to left in the formula (or equivalently on the fermion loop). On the other hand $F_{\mu\nu}^{\mathrm{R}}$ belongs to the class RR, and it does not flip the chiral label. Further, it is observed that in such a chiral invariant expression any two adjacent chiral labels belonging to two different factors are equal (e.g. the label L in $\hat{D}_\mu m_{\mathrm{RL}}\hat{D}_\nu m_{\mathrm{LR}}$). This must be so in order to preserve covariance under chiral transformations. Moreover, if the expression is inside the trace, the first and last chiral labels must also coincide for the same reason, due to the cyclic property of the trace. Thus in chiral covariant expressions the following convention can be used (see Sect. 2 of [1] for further details):

Convention 2. In expressions where the chiral labels are combined preserving chirality, these labels are redundant and will be suppressed, so a term such as $X_{\mathrm{RR}}Y_{\mathrm{RL}}Z_{\mathrm{LR}}$ will be written as $(\mathsf{XYZ})_{\mathrm{RR}}$[1]. Inside a trace it is sufficient to write $\mathrm{tr}(\mathsf{XYZ})$ plus the convention that the first (and last) implicit label is R. (This latter convention is needed to fix the sign in the pseudo-parity odd case.)

For instance

$$\begin{aligned}\mathrm{tr}(\mathsf{F}_{\mu\nu}\hat{\mathsf{D}}_\mu\mathsf{m}\hat{\mathsf{D}}_\nu\mathsf{m}) &= \mathrm{tr}(F_{\mu\nu}^{\mathrm{R}}\hat{D}_\mu m_{\mathrm{RL}}\hat{D}_\nu m_{\mathrm{LR}})\\ &= \pm\mathrm{tr}(F_{\mu\nu}^{\mathrm{L}}\hat{D}_\mu m_{\mathrm{LR}}\hat{D}_\nu m_{\mathrm{RL}})\\ &= \frac{1}{2}\mathrm{tr}(F_{\mu\nu}^{\mathrm{R}}\hat{D}_\mu m_{\mathrm{RL}}\hat{D}_\nu m_{\mathrm{LR}})\\ &\pm \frac{1}{2}\mathrm{tr}(F_{\mu\nu}^{\mathrm{L}}\hat{D}_\mu m_{\mathrm{LR}}\hat{D}_\nu m_{\mathrm{RL}}). \qquad (4)\end{aligned}$$

The $\pm$ refer to $W^\pm$, respectively.

In the pseudo-parity even sector, the cyclic property of the trace works as usual within the index-free notation introduced by Convention 2 [1]. However, the cyclic property is modified for W^-. Let X be of type LR or RL then

$$\begin{aligned}\mathrm{tr}(\mathsf{Xm}) &= \mathrm{tr}(X_{\mathrm{RL}}m_{\mathrm{LR}}) = \mathrm{tr}(m_{\mathrm{LR}}X_{\mathrm{RL}}) = \pm\mathrm{tr}(m_{\mathrm{RL}}X_{\mathrm{LR}})\\ &= \pm\mathrm{tr}(\mathsf{mX}), \quad \text{in } W^\pm. \qquad (5)\end{aligned}$$

This is equivalent to saying that, in W^-, the object m changes sign under the cyclic property. The same is true for any object that flips the chiral label, i.e. of the type RL or LR. Consider now the following identities in W^- (where f and g are ordinary functions):

$$\mathrm{tr}(f(\mathsf{m})g(\mathsf{m})) = \mathrm{tr}(g(\mathsf{m})f(-\mathsf{m})) = \mathrm{tr}(g(-\mathsf{m})f(\mathsf{m})). \quad (6)$$

The first equality follows from moving $f(\mathsf{m})$ to the right, the second one from moving $g(\mathsf{m})$ to the left using the (modified) cyclic property. This equality implies that only the even component of the function $f(x)g(x)$ (under $x \to -x$) contributes. This is just an illustration of the obvious consistency condition stating that the number of chirality

[1] This example assumes, of course, that we know beforehand that Y and Z flip the chirality label and X does not. This is the case in practice since $W^\pm$ is constructed with $D_\mu^{\mathrm{R,L}}$, m_{LR} and m_{RL}.

flipping factors must always be even (e.g. $f(\mathsf{m})g(\mathsf{m})$ must contain even powers of m only) because in any expression inside the trace the first and the last chiral labels must coincide due to chiral invariance. This observation applies to W^+ as well.

In this notation, the chiral rotations $m_{\mathrm{LR}} \to \Omega_{\mathrm{L}}^{-1} m_{\mathrm{LR}} \Omega_{\mathrm{R}}$, etc., become

$$\mathsf{m} \to \Omega^{-1}\mathsf{m}\Omega, \quad \mathsf{v}_\mu \to \Omega^{-1}\mathsf{v}_\mu\Omega + \Omega^{-1}\partial_\mu\Omega, \tag{7}$$

whereas for infinitesimal rotations, $\Omega_{\mathrm{R,L}} = \exp(\alpha_{\mathrm{R,L}})$ with $\alpha_{\mathrm{R,L}}$ infinitesimal:

$$\delta\mathsf{m} = [\mathsf{m}, \alpha], \quad \delta\mathsf{v}_\mu = \hat{\mathsf{D}}_\mu\alpha. \tag{8}$$

A further convention is introduced in [1], namely

Convention 3. In an expression $f(A_1, B_2, \ldots)XY\cdots$ the ordering labels $1, 2, \ldots$ will denote the actual position of the operators $A, B, \ldots$ relative the fixed elements $X, Y, \ldots$ so that A is to be placed before X, B between X and Y, etc. That is, for a separable function $f(a, b, \ldots) = \alpha(a)\beta(b)\cdots$, the expression stands for $\alpha(A)X\beta(B)Y\cdots$

Note that this convention is independent of Conventions 1 and 2. Combining the several conventions and the cyclic property one has, for instance,

$$\begin{aligned}\mathrm{tr}(f(\mathsf{m}_1, \mathsf{m}_2)\mathsf{F}_{\mu\nu}\mathsf{F}_{\alpha\beta}) &= \mathrm{tr}(f(-\mathsf{m}_3, \mathsf{m}_2)\mathsf{F}_{\mu\nu}\mathsf{F}_{\alpha\beta}) \\ &= \mathrm{tr}(f(-\mathsf{m}_2, \mathsf{m}_1)\mathsf{F}_{\alpha\beta}\mathsf{F}_{\mu\nu})\end{aligned} \tag{9}$$

in W^-. In the first equality, m_1 (m in position 1) is moved to position 3 (i.e. becomes the rightmost factor) using the cyclic property, becoming $-\mathsf{m}_3$. Then, in the second equality $\mathsf{F}_{\mu\nu}$ is moved to the rightmost position, and the position labels of m are modified accordingly.

Before proceeding, let us comment on the meaning of an expression, such as $f(A_1, B_2, C_3)XY$, with operators acting in different positions. It should be clear that such an operator is a well-defined one. The simplest way to reduce it to a more usual form is by expressing the function f as a linear combination of separable functions,

$$f(z_1, z_2, z_3) = \sum_i \alpha_i(z_1)\beta_i(z_2)\gamma_i(z_3); \tag{10}$$

then

$$f(A_1, B_2, C_3)XY = \sum_i \alpha_i(A)X\beta_i(B)Y\gamma_i(C), \tag{11}$$

and the right-hand side is perfectly well defined. In fact, such a representation in terms of separable functions is the usual means by which the Convention 3 enters in the calculations (typically the sum over i corresponds to an integration over the momentum of the loop). An alternative method to fully characterize the operator $f(A_1, B_2, C_3)XY$ is by means of its matrix elements. In this context, the natural procedure is to use as a basis the ones formed by the eigenvectors of the operators A, B and C. Let us denote these basis elements by $|n, A\rangle$, $|m, B\rangle$ and $|r, C\rangle$, with associated eigenvalues a_n, b_m and c_r, and let $\langle n, A|$, $\langle m, B|$ and $\langle r, C|$ be the corresponding dual basis; then

$$\langle n, A|f(A_1, B_2, C_3)XY|r, C\rangle = \sum_m f(a_n, b_m, c_r)X_{nm}Y_{mr}, \tag{12}$$

where $X_{nm} = \langle n, A|X|m, B\rangle$ and $Y_{mr} = \langle m, B|Y|r, C\rangle$. (This is easily established using the previous representation in terms of separable functions.) This kind of representation is in fact the one usually employed in the literature (see e.g. [11]). The point to be emphasized is that the operator depends solely on the function f itself and not on any particular representation.

Special care requires the use of the Convention 3 in combination with Conventions 1 and 2 in practical applications. This is because the meaning of the symbols under the Convention 2, depends on its position in the formula. For instance, in an expression such as $\mathrm{tr}(f(\mathsf{m}_1, \mathsf{m}_2)\mathsf{F}_{\mu\nu}\mathsf{F}_{\alpha\beta})$, where f is a complicated function it may not be clear how to expand the formula, i.e., how to put back the chiral labels. Fortunately, there is a simple general procedure to do so, namely, to decompose f into its even and odd components under $\mathsf{m}_{1,2} \to \pm\mathsf{m}_{1,2}$:

$$\begin{aligned}f(\mathsf{m}_1, \mathsf{m}_2) = A(\mathsf{m}_1^2, \mathsf{m}_2^2) + \mathsf{m}_1 B(\mathsf{m}_1^2, \mathsf{m}_2^2) \\ + \mathsf{m}_2 C(\mathsf{m}_1^2, \mathsf{m}_2^2) + \mathsf{m}_1\mathsf{m}_2 D(\mathsf{m}_1^2, \mathsf{m}_2^2).\end{aligned} \tag{13}$$

As noted above, consistency requires f to be even under $\mathsf{m} \to -\mathsf{m}$; thus $B = C = 0$. This yields

$$\begin{aligned}&\mathrm{tr}[f(\mathsf{m}_1, \mathsf{m}_2)\mathsf{F}_{\mu\nu}\mathsf{F}_{\alpha\beta}] \\ &\quad = \mathrm{tr}[A(\mathsf{m}_1^2, \mathsf{m}_2^2)(\mathsf{F}_{\mu\nu})(\mathsf{F}_{\alpha\beta})] \\ &\qquad + \mathrm{tr}[D(\mathsf{m}_1^2, \mathsf{m}_2^2)(\mathsf{m}\mathsf{F}_{\mu\nu})(\mathsf{m}\mathsf{F}_{\alpha\beta})].\end{aligned} \tag{14}$$

Now, from our conventions it unambiguously follows that the chiral labeling is

$$\begin{aligned}&\mathrm{tr}[A(m_{R1}^2, m_{R2}^2)(F_{\mu\nu}^{\mathrm{R}})(F_{\alpha\beta}^{\mathrm{R}})] \\ &\quad + \mathrm{tr}[D(m_{R1}^2, m_{L2}^2)(m_{\mathrm{RL}}F_{\mu\nu}^{\mathrm{L}})(m_{\mathrm{LR}}F_{\alpha\beta}^{\mathrm{R}})].\end{aligned} \tag{15}$$

In this formula Convention 1 still applies, $m_{\mathrm{R}}^2 = m_{\mathrm{RL}}m_{\mathrm{LR}}$ and $m_{\mathrm{L}}^2 = m_{\mathrm{LR}}m_{\mathrm{RL}}$[2].

Several illustrations of the Convention 3 (besides its use in W^+) have been presented in [1] and elsewhere [17, 18]. Here we present another application which will be needed below. First, let us introduce standard differential geometry notation: the quantities $\mathrm{d}x_\mu$ are anticommuting, $\mathrm{d}^d x = \mathrm{d}x_0\mathrm{d}x_1\cdots\mathrm{d}x_{d-1}$, d is the differential operator $\mathrm{d}x_\mu\partial_\mu$, v stands for $\mathsf{v}_\mu\mathrm{d}x_\mu$, $\mathsf{D} = \mathsf{D}_\mu\mathrm{d}x_\mu$, $\mathsf{F} = \mathsf{D}^2 = \mathrm{d}\mathsf{v} + \mathsf{v}^2$, etc. Consider now the following n-form

$$X = f(A_1, \ldots, A_n)(\mathrm{d}A)^n, \tag{16}$$

where A is some matrix-valued function defined on some manifold, and $f(z_1, \ldots, z_n)$ is an ordinary function. We want to compute $\mathrm{d}X$. To this end, recall the rule [1]

$$\delta f(A) = \frac{f(A_1) - f(A_2)}{A_1 - A_2}\delta A, \tag{17}$$

[2] Alternatively, the second term could have been written as $\mathrm{tr}[D(\mathsf{m}_1^2, \mathsf{m}_2^2)(\mathsf{m}\mathsf{F}_{\mu\nu}\mathsf{m})(\mathsf{F}_{\alpha\beta})]$, yielding $\mathrm{tr}[D(m_{R1}^2, m_{R2}^2)(m_{\mathrm{RL}}F_{\mu\nu}^{\mathrm{L}}m_{\mathrm{LR}})(F_{\alpha\beta}^{\mathrm{R}})]$. This is equivalent to the previous result.

for an arbitrary variation of A (the labels 1 and 2 refer to A before and after δA, respectively, following Convention 3). In particular, $\mathrm{d}f(A) = (f(A_1) - f(A_2))/(A_1 - A_2)\mathrm{d}A$. Applying the operator d to X as defined in (16), and using the previous rule to variate each of the arguments A_i in X, immediately yields

$$\mathrm{d}X = \Delta f(A_1, \ldots, A_{n+1})(\mathrm{d}A)^{n+1}, \quad (18)$$

with

$$\begin{aligned}&\Delta f(z_1, \ldots, z_{n+1})\\ &= \sum_{k=1}^{n}(-1)^{k+1}\Big\{\Big(f(z_1, \ldots, z_k, z_{k+2}, \ldots, z_{n+1})\\ &-f(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_{n+1})\Big)\Big/\Big(z_k - z_{k+1}\Big)\Big\}. \quad (19)\end{aligned}$$

Because the operator Δ is a representation of the operator d acting in the space of ordinary functions, it follows that $\Delta^2 = 0$, as is readily verified. The same operator appears when the covariant derivative $\hat{\mathsf{D}}$ is used, instead of d, although in this case terms involving the field strength tensor F are also generated.

2.2 Application to the Wess–Zumino–Witten action

An interesting illustration of the usefulness of the Convention 3 can be given by using it to explicitly integrate the WZW action [9]. In two spacetime dimensions, the WZW functional takes the form

$$\Gamma_{\mathrm{WZW}}[U] = \frac{\eta_2}{4\pi}\int_{B_3}\Omega_3, \quad \Omega_3 = -\frac{1}{3}\mathrm{tr}\left[(U^{-1}\mathrm{d}U)^3\right]. \quad (20)$$

The integration takes place in the interior of a three-dimensional ball with a sphere S^2 (the compactified spacetime) as boundary. The field $U(x,t)$, which takes values on some matrix group, interpolates between $u(x)$, at $t = 1$ and a single point, say $U = 1$, at $t = 0$. Because Ω_3 is a closed 3-form and $u(x)$ contractile, the functional can be written as the integral of a 2-form over S^2:

$$\Gamma_{\mathrm{WZW}}[U] = \frac{\eta_2}{4\pi}\int_{S^2}\Omega_2, \quad (21)$$

with

$$\Omega_2 = -\int_0^1 \mathrm{d}t\mathrm{tr}\left[(U^{-1}\partial_t U)(U^{-1}\mathrm{d}U)^2\right], \quad (22)$$

and the result does not depend on the concrete interpolation. We will make use of our Convention 3 in order to explicitly carry out the integration on the parameter t. As interpolating field, let us take[3] $U(x,t) = u(x)^t = \exp(t\log u(x))$. The branch of the logarithm can be chosen with continuity because $u(x)$ is contractile to 1, by assumption. Using Convention 3

$$U^{-1}\partial_t U = \log u, \quad U^{-1}\mathrm{d}U = \frac{1-(u_2/u_1)^t}{u_1 - u_2}\mathrm{d}u, \quad (23)$$

where the labels 1 and 2 refer to before and after application of $\mathrm{d}u$, respectively. When these formulas are inserted in the expression of Ω_2, $U^{-1}\partial_t U$ carries the position label 1, the first $U^{-1}\mathrm{d}U$ block gives rise to the labels 1 and 2, and the second block to labels 2 and 3. Due to the cyclic property u_3 is then identified with u_1[4]. This gives

$$\Omega_2 = -\int_0^1 \mathrm{d}t\mathrm{tr}\left[\log u_1 \frac{1-(u_2/u_1)^t}{u_1 - u_2}\frac{1-(u_1/u_2)^t}{u_2 - u_1}\mathrm{d}u^2\right]. \quad (24)$$

The point of following this procedure is that the dependence on t is now explicit and u_1 and u_2 are effectively c-numbers; therefore the integration over t is immediate:

$$\Omega_2 = \mathrm{tr}\left[h_{\mathrm{WZW}}(u_1, u_2)\mathrm{d}u^2\right], \quad (25)$$

where the function h_{WZW} is given by

$$\begin{aligned}h_{\mathrm{WZW}}(z_1, z_2) = \frac{1}{z_1 - z_2}&\left(\frac{\log(z_1) - \log(z_2)}{z_1 - z_2}\right.\\ &\left.- \frac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\right). \quad (26)\end{aligned}$$

It should be noted that, due to the cyclic property, the relation in (25) does not uniquely determine $h_{\mathrm{WZW}}(z_1, z_2)$ unless the further constraint $h_{\mathrm{WZW}}(z_1, z_2) = -h_{\mathrm{WZW}}(z_2, z_1)$ is imposed. On top of this, a symmetric component can be added which does not contribute inside the trace. In actual applications of the formula the purely antisymmetric version of h_{WZW} is clearly preferred, since an unsymmetrized function (although not Ω_2 itself) could in general present spurious singularities at $u_1 = u_2$ as well as spurious scale violations. The latter refers to the following. Ω_3 is invariant under an arbitrary local rescaling of U, $U(x,t) \to \lambda(x,t)U(x,t)$, where λ is a c-number. Because Ω_2 is unique in some sense (to be discussed below), it must also display this invariance. The invariance under a global rescaling already implies that (the symmetrized version of) h_{WZW} must be a homogeneous function; the possible breaking introduced by the logarithm is canceled in this version. Further, the invariance under a local rescaling is also preserved due to $h_{\mathrm{WZW}}(z,z) = 0$ in the antisymmetric version.

The precise statement is that h_{WZW} is the unique function that works for a generic gauge group, that is, if no further assumptions are made on the algebraic properties of the field $u(x)$. For instance, any function h would give

[3] The choice $U(x,t) = 1 + t(u(x) - 1)$ is even simpler and, of course, gives the same result, however, it may be disturbing that it does not lie on the group manifold when $u(x)$ belongs to a group of unitary matrices.

[4] No confusion should arise with our previous observation (cf. (5)) that m changes sign under the cyclic property in the case of W^-, since Conventions 1 and 2 are not being used here. On the other hand, $\mathrm{d}u$ does change sign in Ω_2 under the cyclic property since it is a one-form.

the correct vanishing result in the particular case of an Abelian gauge group. Another interesting case is that of $u \in \mathrm{SU}(2)$. For this group $u + u^{-1}$ is c-number and the same goes for any function $g(u)$ such that $g(z) = g(z^{-1})$. This is sufficient to show that, for any antisymmetric function $h(z_1, z_2)$,

$$\mathrm{tr}\left[h(u_1,u_2)(u^{-1}\mathrm{d}u)^2\right] = \mathrm{tr}\left[h(u,u^{-1})(u^{-1}\mathrm{d}u)^2\right] \quad (u \in \mathrm{SU}(2)); \tag{27}$$

thus, in particular

$$\Omega_2 = \mathrm{tr}\left[h_{\mathrm{WZW}}(u)(u^{-1}\mathrm{d}u)^2\right] \quad (u \in \mathrm{SU}(2)), \tag{28}$$

with

$$h_{\mathrm{WZW}}(z) = \frac{4\log(z) - z^2 + z^{-2}}{2(z - z^{-1})^2}. \tag{29}$$

It is interesting to note that, in principle, the function $h_{\mathrm{WZW}}(z_1, z_2)$ can also be determined through an equation involving only ordinary functions and no differential forms. This comes about as follows. The equation to be solved is $\Omega_3 = \mathrm{d}\Omega_2$, where Ω_2 is the unknown. For the latter, the general form in (25) is proposed, whereas Ω_3 can be rewritten as

$$\Omega_3 = \mathrm{tr}\left[-\frac{1}{3}\frac{1}{U_1U_2U_3}\mathrm{d}U^3\right]. \tag{30}$$

The relation in (18) then implies

$$-\frac{1}{3}\frac{1}{z_1z_2z_3} = \frac{1}{3}\left(\Delta h_{\mathrm{WZW}}(z_1,z_2,z_3) + \Delta h_{\mathrm{WZW}}(z_2,z_3,z_1) + \Delta h_{\mathrm{WZW}}(z_3,z_1,z_2)\right), \tag{31}$$

where the cyclic property has been used to be able to equate both sides of the equation[5]. Because of the lack of the appropriate mathematical techniques, this kind of equation does not seem to be particularly useful to determine the function h_{WZW}; nevertheless it has the merit of reducing a problem of differential forms to one of ordinary functions. Certainly it serves to check our previous result for h_{WZW}.

The analogous expressions in four dimensions are

$$\Gamma_{\mathrm{WZW}}[U] = \frac{\eta_4}{48\pi^2}\int_{B_5}\Omega_5, \qquad \Omega_5 = -\frac{1}{5}\mathrm{tr}\left[(U^{-1}\mathrm{d}U)^5\right]. \tag{32}$$

$$\Omega_4 = \mathrm{tr}\left[\left(\frac{1}{u_{12}u_{23}u_{34}u_{41}}\left(\frac{u_1}{u_2} + \frac{u_1}{u_3} + \frac{u_1}{u_4} - \frac{1}{2}\frac{u_1u_3}{u_2u_4}\right) + 2\frac{u_{12} - u_{41}}{u_{12}^2u_{13}u_{41}^2}\log(u_1)\right)\mathrm{d}u^4\right], \tag{33}$$

where $u_{ij} = u_i - u_j$. For the sake of brevity the function has not been explicitly symmetrized in order to extract its invariant component under cyclic permutations. As noted before in the two-dimensional case, such a symmetrization is needed in practice.

Because in the four-dimensional case the integral refers to a 4-form, the formula seems to predict a vanishing value (or more generally, a multiple of $2\pi\mathrm{i}$) for the WZW term when the gauge group is three-dimensional such as SU(2), whereas actually the result is a multiple of $\mathrm{i}\pi$. However, the $\mathrm{i}\pi$ result corresponds to configurations which cannot be contracted within SU(2). Another observation is that the use of arbitrary functions $h(z_1, z_2)$ in two dimensions, or $h(z_1, z_2, z_3, z_4)$ in four dimensions, allows one to propose phenomenological contributions to the effective action in the pseudo-parity odd sector, which are more general than the usual WZW term. All these possible new contributions are automatically invariant under global vector transformations ($u \mapsto \Omega^{-1}u\Omega$ with constant Ω). Among them, the WZW term is singularized because it is invariant under global chiral transformations ($u \mapsto \Omega_{\mathrm{L}}^{-1}u\Omega_{\mathrm{R}}$ with constant $\Omega_{\mathrm{L,R}}$). On the other hand, it can be noted that Ω_2 (or Ω_4 in four dimensions) is not the unique solution of $\Omega_3 = \mathrm{d}\Omega_2$, since $\Omega_2 + \mathrm{d}\omega$ (ω being an arbitrary 1-form) would also be a solution. Ω_2 is singularized because it is the one solution which is manifestly invariant under global vector transformations.

3 Explicit computation of the covariant current

As stated above, our purpose is to compute the leading term of the pseudo-parity odd component of the effective action of Dirac fermions. By leading term we mean the one with the smaller number of covariant derivatives, and covariant will always refer to chiral gauge transformations. Because $W^-[v,m]$ contains the Levi-Civita pseudo-tensor, the leading term is that with d Lorentz indices, d being the spacetime dimension, which is assumed to be even. In practice we will consider $d = 0, 2, 4$. All other higher order terms in the derivative expansion are ultraviolet finite and thus free from anomalies and multivaluation. The chiral anomaly, multivaluation and topological pieces of the effective action are contained in the leading term. There are no other anomalies (such as scale or parity anomalies) in $W^-[v,m]$ in even dimensions. Since no higher orders will be considered in this work, from now on $W^-[v,m]$ will be used to refer to the leading term. We will always work with the LR version of the effective action except in Sect. 5.3 and Appendix A.

3.1 The covariant current

Due to the presence of the chiral anomaly in the pseudo-parity odd component of the effective action, $W^-[v,m]$ is not a chiral invariant functional, and this makes it advisable to use an indirect procedure to compute it. We will

[5] It should be noted that the operator Δ does not commute with the operation P of projecting the component which is invariant under cyclic permutations. Thus, if Δ is now applied to the right-hand side of (31) the result does not vanish (despite the property $\Delta^2 = 0$) but it does vanish after a subsequent application of P. This expresses the fact that the 3-form Ω_3 is closed.

adopt the traditional Schwinger approach [15] of working with the current, i.e. the variation of the effective action [13,11]. The reason of course is that there is a version of the current which is chiral covariant and thus easier to treat.

For subsequent reference, we note that there are two quantities to be distinguished: the effective "current" J_v^- which is related to the variation with respect to the gauge fields, and the effective "density" J_m^- which is the variation with respect to the scalar fields. The consistent effective current and density will be defined as

$$\delta W^-[v,m] = \left\langle \mathsf{J}_v^- \delta\mathsf{v} + \mathsf{J}_m^- \delta\mathsf{m} \right\rangle . \tag{34}$$

Our conventions 1 and 2 are being used, $\delta\mathsf{m}$ and $\delta\mathsf{v}$ are arbitrary variations of the external fields, and $\delta\mathsf{m}$, $\delta\mathsf{v}$, J_v^- and J_m^- are 0-, 1-, $(d-1)$- and d-forms, respectively. $\langle\ \rangle$ was defined in (3).

In addition, one has to distinguish between the consistent and the covariant currents. The former is the variation of the effective action, but it fails to be chiral covariant due to the presence of the chiral anomaly. On the other hand, the chiral covariant version $\mathsf{J}_{v,\mathrm{c}}^-$ is not consistent, i.e., is not a true variation. Both versions of the effective current are realizations of the same formal object. This means that they coincide in their ultraviolet finite pieces and so they differ only by a counterterm which is a polynomial in the external fields and their derivatives:

$$\mathsf{J}_v^- = \mathsf{J}_{v,\mathrm{c}}^- + \mathsf{P}(v). \tag{35}$$

$\mathsf{P}(v)$ is a fixed known polynomial which depends solely on the gauge fields and its derivatives. This polynomial is purely geometrical in the same sense as the chiral anomaly, and in fact is completely determined by the anomaly [10].

The idea of the calculation of $W^-[v,m]$ is as follows. We will explicitly compute the covariant current, then we will write the most general form $W^-[v,m]$ consistent with chiral and Lorentz symmetries, with some functions as unknowns, and finally these unknowns will be chosen so as to reproduce the current. It is only necessary to make sure that the effective action is uniquely determined by this procedure. That this will be the case can be seen by the following argument. Let $A^-[v,m]$ denote a possible ambiguity in the effective action allowed by this procedure. Because the current is reproduced, $A^-[v,m]$ must actually be a functional of m only. In addition, $A^-[m]$ must be chiral invariant, since we have already imposed the correct chiral transformation on our functional. Then it can be evaluated in any chirally rotated configuration, and in particular one can always choose $m_{\mathrm{LR}} = m_{\mathrm{RL}}$. It follows that the ambiguity vanishes since this functional is odd under pseudo-parity, i.e., under exchange of the labels L and R.

It is also possible to use the density J_m^- instead of the current. In this case the ambiguity can only be a function of v and it is easily shown that no such chiral invariant functional exists, at leading order. (The previous argument for the current holds, however, to all orders in the derivative expansion.) An advantage of J_m^- would be that there is no distinction between consistent and covariant density (the consistent density is automatically chiral covariant). Nevertheless, within a derivative expansion, the current is preferable for purely technical reasons, namely, the current contains $d-1$ derivatives whereas the density contains d ones, and thus it requires more work. Explicit formulas for the effective density in two and four dimensions are given in Sect. 5.2.

In order to highlight the main results of this section, the calculation itself will be deferred until the end of this section. As will be clear from the calculation below, the general form of the covariant current in two and four dimensions (of course, at leading order in the derivative expansion) is

$$\begin{aligned} \mathsf{J}_{v,\mathrm{c},d=2}^- &= A(\mathsf{m}_1,\mathsf{m}_2)\mathsf{m}', \\ \mathsf{J}_{v,\mathrm{c},d=4}^- &= A(\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3,\mathsf{m}_4)\mathsf{m}'^3 + A(\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3,)\mathsf{F}\mathsf{m}' \\ &\quad + A'(\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3)\mathsf{m}'\mathsf{F}, \end{aligned} \tag{36}$$

where m' denotes the 1-form $\hat{\mathsf{D}}\mathsf{m}$. The subindex c in the currents recalls that this is the covariant current. The various symbols A denote different known functions. We will often use the shorthand notation A_{12} to denote $A(\mathsf{m}_1,\mathsf{m}_2)$, A_{123} to denote $A(\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3)$, etc. In addition, $A_{\underline{1}2}$ will denote $A(-\mathsf{m}_1,\mathsf{m}_2)$, etc.

As we have just mentioned, the formulas in (36) follow from the explicit calculation; nevertheless, by now it is probably already obvious that they are just the most general possible form for the currents at leading order consistent with Lorentz and chiral gauge invariance. Let us see which properties are to be expected for the functions A in $\mathsf{J}_{v,\mathrm{c}}^-$. Because there is no scale anomaly in $W^-[v,m]$, A should homogeneous functions of the appropriate degree. Next, there is the *consistency condition* that in each term of $\mathsf{J}_{v,\mathrm{c}}^-$ there should be as many L labels as R labels, thus m must appear an even number of times:

$$\begin{aligned} A_{12} &= -\,A_{\underline{12}}, \quad A_{123} = -A_{\underline{123}}, \\ A'_{123} &= -\,A'_{\underline{123}}, \quad A_{1234} = -A_{\underline{1234}} \end{aligned} \tag{37}$$

(that is, $A(\mathsf{m}_1,\mathsf{m}_2) = A(-\mathsf{m}_1,-\mathsf{m}_2)$, etc.).

A further condition is implied by the fact that $W^-[v,m]$ is purely imaginary. First, note that the functions A are all purely real since there are no i's in the formulas nor can they be generated during the calculation, except through γ_5 when $d = 4n+2$. The possible factor i is explicit through η_d in the normalization of $\langle\ \rangle$ (cf. (3)). On the other hand, the fact that all quantities involved behave in a well-defined way under Hermitian conjugation allows one to reformulate this conjugation in terms of an equivalent *mirror transformation* which has the advantage of being purely algebraic (no complex conjugation is involved). Such a mirror transformation is defined by the following rules:

(i) the elementary objects m, v (or $\delta\mathsf{v}$) and D are mirror invariant,
(ii) the transformation is linear, and
(iii) the order of the factors is transposed (regardless of whether they are functions or differential forms). The

transformation of derived quantities follows from the previous rules, thus $\mathsf{F} \to \mathsf{F}$, $\mathsf{dm} \to -\mathsf{dm}$, $\mathsf{dv} \to \mathsf{dv}$, $\hat{\mathsf{D}} \to \mp\hat{\mathsf{D}}$ (depending on whether it acts commuting or anticommuting, respectively), $\mathsf{m}' \to -\mathsf{m}'$, etc. For instance,

$$\begin{aligned} &\langle \mathsf{mvm}^{-1}\mathsf{v}\rangle \to \langle \mathsf{vm}^{-1}\mathsf{vm}\rangle = -\langle \mathsf{mvm}^{-1}\mathsf{v}\rangle, \\ &A(\mathsf{m}_1,\mathsf{m}_2)\mathsf{m}' \to -A(\mathsf{m}_2,\mathsf{m}_1)\mathsf{m}'. \end{aligned} \tag{38}$$

The antihermiticity of $W^-[v,m]$ implies that this quantity is odd under the mirror transformation whereas J_v^- and J_m^- are even. Therefore, the following conditions are found:

$$A_{12} = -A_{21}, \quad A'_{123} = -A_{321}, \quad A_{1234} = -A_{4321}. \tag{39}$$

Thus the function A'_{123} is not independent.

Finally, there is an extremely important property satisfied by these functions which is finiteness in the coincidence limit. This refers to the following. The most general form of A_{12} allowed by consistency and mirror symmetry, is

$$A_{12} = \mathsf{m}_1 f(\mathsf{m}_1^2,\mathsf{m}_2^2) - \mathsf{m}_2 f(\mathsf{m}_2^2,\mathsf{m}_1^2), \tag{40}$$

for certain function f. Inserting this general form in the expression of $\mathsf{J}_{c,v}^-$ in (36) and making explicit the chiral labels yields

$$(\mathsf{J}_{v,c}^-)_{\mathrm{R}} = f(\mathsf{m}_{\mathrm{R}1}^2,\mathsf{m}_{\mathrm{R}2}^2)(\mathsf{mm}')_{\mathrm{R}} - f(\mathsf{m}_{\mathrm{R}1}^2,\mathsf{m}_{\mathrm{R}2}^2)(\mathsf{m}'\mathsf{m})_{\mathrm{R}}. \tag{41}$$

As noted before, in order to numerically evaluate this expression, a natural procedure is to use a basis of eigenvectors of m^2 [1]. In this way, $(\mathsf{mm}')_{\mathrm{R}}$ and $(\mathsf{m}'\mathsf{m})_{\mathrm{R}}$ are replaced by matrix elements, whereas the m^2 are replaced by eigenvalues. In particular, in the diagonal matrix elements, m_1^2 and m_2^2 take the same value (note that $m_{\mathrm{LR}}m_{\mathrm{RL}}$ and $m_{\mathrm{RL}}m_{\mathrm{LR}}$ are related by a similarity transformation and thus they have the same eigenvalues). Finiteness of the current requires that f must be finite as its two arguments coincide. (Because the terms with mm' and $\mathsf{m}'\mathsf{m}$ have different chiral labels no cancellation can take place among them in general.) In summary, the functions A_{12}, A_{123}, etc., must be regular as two or more arguments coincide up to a sign. This is automatically satisfied by the true functional describing the current, as no physical singularity exists in the coincidence limit (cf. (68) and (69) below); however, the formalism allows one to write functionals which violate this condition. Such functionals are only formal and are meaningless or at least ambiguous. On the other hand, physical singularities can occur as $\mathsf{m} \to 0$ and they will be reflected in the effective action and currents.

After this discussion, let us quote the result coming from the explicit calculation in two dimensions:

$$\begin{aligned} A_{12} = &-\frac{2}{\mathsf{m}_1-\mathsf{m}_2} \\ &+ \frac{2\mathsf{m}_1\mathsf{m}_2}{(\mathsf{m}_1-\mathsf{m}_2)(\mathsf{m}_1^2-\mathsf{m}_2^2)}\log(\mathsf{m}_1^2/\mathsf{m}_2^2). \end{aligned} \tag{42}$$

The corresponding four-dimensional formulas are collected in Appendix B. It can be checked that the full functions have all the expected properties, and in particular they preserve scale invariance and are regular in the coincidence limits. At this point, it is perhaps worth noticing another essential property of these functions, namely, they are unambiguous. The analogous functions for the effective action are not unique due to integration by parts and the trace cyclic property. This not the case for the current; these functions are the unique result of the calculation. The formulas can only be simplified by considering particular cases, i.e., particular flavor groups.

Analyzing the form of the functions A, we have found it convenient to introduce the auxiliary functions $\bar{A}$:

$$A_{12} = \bar{A}_{1\underline{2}}, \quad A_{123} = \bar{A}_{12\underline{3}}, \quad A_{1234} = \bar{A}_{1\underline{2}3\underline{4}} \tag{43}$$

(i.e. $A(\mathsf{m}_1,\mathsf{m}_2) = \bar{A}(\mathsf{m}_1,-\mathsf{m}_2)$, etc.). For these functions, consistency and mirror symmetry translates into

$$\begin{aligned} &\bar{A}_{12} = -\bar{A}_{\underline{12}}, \quad \bar{A}_{123} = -\bar{A}_{\underline{123}}, \quad \bar{A}_{1234} = -\bar{A}_{\underline{1234}}, \\ &\bar{A}_{12} = \bar{A}_{21}, \quad \bar{A}_{1234} = \bar{A}_{4321}. \end{aligned} \tag{44}$$

As we will see later, further conditions are implied by the fact that the underlying theory is Lorentz and chiral covariant. In particular, this implies

$$\bar{A}_{123} = \bar{A}_{132}. \tag{45}$$

Remarkably, the true functions $\bar{A}$, i.e. those resulting from the calculation, in four dimensions turn out to have a larger symmetry, namely, they are completely symmetric functions of their arguments:

$$\begin{aligned} &\bar{A}_{12} = \bar{A}_{21}, \quad \bar{A}_{123} = \bar{A}_{213} = \bar{A}_{231}, \\ &\bar{A}_{1234} = \bar{A}_{2134} = \bar{A}_{2341}. \end{aligned} \tag{46}$$

(The complete symmetry also holds in two dimensions but it this case this follows from previous symmetries.) It is not clear why, in the four-dimensional case, the symmetry is larger than expected. This symmetry does not follow from Lorentz invariance and (anomalous) chiral symmetry, since it is possible to write Lorentz invariant functionals with the correct chiral anomaly but with associated variations which are not symmetric functions under permutation of their arguments (see (90) below). It seems to be a property of the true current only. The same symmetry is also found for the effective density in two and four dimensions (see Sect. 5.2).

3.2 Explicit computation of the covariant current

Let us consider a first order variation of the effective action. This is formally given by

$$\delta W[v,m] = -\mathrm{Tr}\left(\frac{1}{\mathrm{D}}\delta\mathrm{D}\right). \tag{47}$$

The variation of the Dirac operator is

$$\delta\mathrm{D} = P_{\mathrm{R}}\delta m_{\mathrm{LR}}P_{\mathrm{R}} + P_{\mathrm{R}}\delta \not{v}_{\mathrm{L}}\, P_{\mathrm{L}} + P_{\mathrm{L}}\delta\not{v}_{\mathrm{R}}\, P_{\mathrm{R}} + P_{\mathrm{L}}\delta m_{\mathrm{RL}}P_{\mathrm{L}}. \tag{48}$$

On the other hand, the inverse Dirac operator can be written as follows. First let us write the Dirac operator as

$$\mathrm{D} = \begin{pmatrix} m_{\mathrm{LR}} & \not{D}_{\mathrm{L}} \\ \not{D}_{\mathrm{R}} & m_{\mathrm{RL}} \end{pmatrix} \tag{49}$$

(where actually the γ_μ stand for submatrices of half dimension after restriction to the LR of RL sectors), and then use the matrix identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & (C-DB^{-1}A)^{-1} \\ (B-AC^{-1}D)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}, \tag{50}$$

where A, B, C, D are square submatrices. (This formula can be rewritten in a way that holds too when A and D have different dimensions and thus B and C are not square matrices.) Now we have

$$\begin{aligned} \mathrm{D}^{-1} &= P_{\mathrm{R}}(m_{\mathrm{LR}} - \not{D}_{\mathrm{L}}\, m_{\mathrm{RL}}^{-1}\, \not{D}_{\mathrm{R}})^{-1} P_{\mathrm{R}} \\ &+ P_{\mathrm{R}}(\not{D}_{\mathrm{R}} - m_{\mathrm{RL}}\, \not{D}_{\mathrm{L}}^{-1}\, m_{\mathrm{LR}})^{-1} P_{\mathrm{L}} \\ &+ P_{\mathrm{L}}(\not{D}_{\mathrm{L}} - m_{\mathrm{LR}}\, \not{D}_{\mathrm{R}}^{-1}\, m_{\mathrm{RL}})^{-1} P_{\mathrm{R}} \\ &+ P_{\mathrm{L}}(m_{\mathrm{RL}} - \not{D}_{\mathrm{R}}\, m_{\mathrm{LR}}^{-1}\, \not{D}_{\mathrm{L}})^{-1} P_{\mathrm{L}}. \end{aligned} \tag{51}$$

Therefore, the variation of the effective action is

$$\begin{aligned} \delta W[v,m] = -\mathrm{Tr}\Big[& P_{\mathrm{R}}(m_{\mathrm{LR}} - \not{D}_{\mathrm{L}}\, m_{\mathrm{RL}}^{-1}\, \not{D}_{\mathrm{R}})^{-1}\delta m_{\mathrm{LR}} \\ &+ P_{\mathrm{R}}(\not{D}_{\mathrm{R}} - m_{\mathrm{RL}}\, \not{D}_{\mathrm{L}}^{-1}\, m_{\mathrm{LR}})^{-1}\delta \not{v}_{\mathrm{R}} \\ &+ P_{\mathrm{L}}(m_{\mathrm{RL}} - \not{D}_{\mathrm{R}}\, m_{\mathrm{LR}}^{-1}\, \not{D}_{\mathrm{L}})^{-1}\delta m_{\mathrm{RL}} \\ &+ P_{\mathrm{L}}(\not{D}_{\mathrm{L}} - m_{\mathrm{LR}}\, \not{D}_{\mathrm{R}}^{-1}\, m_{\mathrm{RL}})^{-1}\delta \not{v}_{\mathrm{L}} \Big]. \end{aligned} \tag{52}$$

This variation can be separated into its pseudo-parity even (without γ_5) and odd (with γ_5) components. Then, Conventions 1 and 2 can directly be applied and this yields

$$\begin{aligned} \delta W^+[v,m] &= -\mathrm{Tr}\Big[(\mathsf{m} - \not{\mathsf{D}}\,\mathsf{m}^{-1}\,\not{\mathsf{D}})^{-1}\delta\mathsf{m} \\ &\quad + (\not{\mathsf{D}} - \mathsf{m}\,\not{\mathsf{D}}^{-1}\,\mathsf{m})^{-1}\delta\not{\mathsf{v}}\Big], \\ \delta W^-[v,m] &= -\mathrm{Tr}\Big[\gamma_5\Big((\mathsf{m} - \not{\mathsf{D}}\,\mathsf{m}^{-1}\,\not{\mathsf{D}})^{-1}\delta\mathsf{m} \\ &\quad + (\not{\mathsf{D}} - \mathsf{m}\,\not{\mathsf{D}}^{-1}\,\mathsf{m})^{-1}\delta\not{\mathsf{v}}\Big)\Big]. \end{aligned} \tag{53}$$

Once our conventions are used, the variations can be rewritten in the simpler form

$$\begin{aligned} \delta W^+[v,m] &= -\mathrm{Tr}\left[\frac{1}{\not{\mathsf{D}}+\mathsf{m}}(\delta\not{\mathsf{v}}+\delta\mathsf{m})\right], \\ \delta W^-[v,m] &= -\mathrm{Tr}\left[\gamma_5\frac{1}{\not{\mathsf{D}}+\mathsf{m}}(\delta\not{\mathsf{v}}+\delta\mathsf{m})\right]. \end{aligned} \tag{54}$$

(Actually, what enters is

$$\frac{1}{2}\left((\not{\mathsf{D}}+\mathsf{m})^{-1}(\delta\not{\mathsf{v}}+\delta\mathsf{m}) + (\not{\mathsf{D}}-\mathsf{m})^{-1}(\delta\not{\mathsf{v}}-\delta\mathsf{m})\right),$$

but the even component under $\mathsf{m}\to-\mathsf{m}$ is automatically selected by the Dirac trace.)

The variation of the pseudo-parity even component is just

$$\delta W^+[v,m] = -\delta\mathrm{Tr}\log[\not{\mathsf{D}}+\mathsf{m}]. \tag{55}$$

Therefore, in our notation, $W^+[v,m]$ is completely identical to a purely vector-like theory (i.e., one with $v_{\mathrm{R}} = v_{\mathrm{L}}$ and $m_{\mathrm{LR}} = m_{\mathrm{RL}}$), a fact already exploited in [1].

The pseudo-parity odd case is different. $\delta W^-[v,m]$ cannot be expressed as the variation of a functional of the form $\mathrm{Tr}[\gamma_5 f(\mathsf{m},\mathsf{D})]$, since that would not allow for the chiral anomaly. Technically the difference with the pseudo-parity even case comes from the cyclic property which is affected by the presence of γ_5 as well as by the different behavior of m, cf. (5). Thus there is an obstruction to integrate the variation preserving all symmetries [13,11]. No such problem arises if one wants to compute just the current or the density: because one particular operator is distinguished, namely $\delta\mathsf{v}$ or $\delta\mathsf{m}$, the cyclic property is no longer required and the anomalous behavior of m under the cyclic property does not enter.

Comparing with its definition in (34), the current can be formally read off from

$$\delta W^-[v,m] = \int \mathrm{d}^d x \,\mathrm{tr}\left[\delta\mathsf{v}_\mu \langle x|\gamma_5\gamma_\mu\frac{1}{\not{\mathsf{D}}+\mathsf{m}}|x\rangle\right] \quad (\delta\mathsf{m}=0), \tag{56}$$

where the trace includes flavor and Dirac spaces. This is formal because the matrix element on the right-hand side is ultraviolet divergent and needs to be given a meaning through some renormalization procedure. Noting that the cyclic property does not enter and in addition that no γ_5 appears in $(\not{\mathsf{D}}+\mathsf{m})^{-1}$, it follows that the symbols D and m behave algebraically as those of an effective vector-like theory. This allows one to use a regularization prescription preserving the corresponding vector gauge invariance. Such an effective vector gauge invariance amounts to chiral covariance for the operator $(\not{\mathsf{D}}+\mathsf{m})^{-1}$ and therefore this procedure will yield the chiral covariant effective current.

In the particular case of two spacetime dimensions, there is a shortcut. The two-dimensional identity $\gamma_5\gamma_\mu = -\eta_2\epsilon_{\mu\nu}\gamma_\nu$ allows one to relate $\delta W^-[v,m]$ with a variation of $W^+[v,m]$, namely

$$\delta W^-[v,m] = \int \mathrm{d}^2 x\,\mathrm{tr}\left[\eta_2\epsilon_{\mu\nu}\delta\mathsf{v}_\mu\frac{\delta W^+}{\delta\mathsf{v}_\nu}\right]. \tag{57}$$

Use of the result in [1],

$$W^+_{2,2}[v,m] = -\frac{1}{4\pi}\int \mathrm{d}^2x\,\mathrm{tr}\left[\left(\mathsf{m}_1\mathsf{m}_2\frac{\log(\mathsf{m}_1^2/\mathsf{m}_2^2)}{\mathsf{m}_1^2-\mathsf{m}_2^2} - 1\right) \times \frac{(\hat{\mathsf{D}}_\mu\mathsf{m})^2}{(\mathsf{m}_1-\mathsf{m}_2)^2}\right], \tag{58}$$

directly produces

$$\mathsf{J}^-_{v,\mathrm{c}} = \frac{2}{\mathsf{m}_1-\mathsf{m}_2}\left(\mathsf{m}_1\mathsf{m}_2\frac{\log(\mathsf{m}_1^2/\mathsf{m}_2^2)}{\mathsf{m}_1^2-\mathsf{m}_2^2} - 1\right)\hat{\mathsf{D}}\mathsf{m}, \tag{59}$$

where the labels 1 and 2 refer to *before* and *after* $\hat{\mathsf{D}}\mathsf{m}$, respectively. From this formula, one can immediately read off the function A_{12} introduced in (36), and this gives the result quoted in (42). In the two-dimensional case, there is yet another method which yields the effective action directly from the anomaly. This method is explained in Sect. 5.4.

In order to compute $\mathsf{J}^-_{v,\mathrm{c}}$ beyond two dimensions we will use the convenient method introduced by Pletnev and Banin [16]. The method can be briefly summarized as follows: Let $f(\mathsf{m},\mathsf{D})$ be an operator constructed out of m and D_μ. In the usual symbols method (see e.g. [14]),

$$\langle x|f(\mathsf{m},\mathsf{D})|x\rangle = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\langle x|f(\mathsf{m},\mathsf{D}+p)|0\rangle, \qquad (60)$$

where $|0\rangle$ is the state with zero wavenumber, i.e. $\langle x|0\rangle = 1$, and the momentum p_μ is just a c-number[6]. The matrix element $\langle x|f(\mathsf{m},\mathsf{D})|x\rangle$ is manifestly gauge covariant; however, $\langle x|f(\mathsf{m},\mathsf{D}+p)|0\rangle$ is not, because of $|0\rangle$. Gauge invariance is recovered only after momentum integration. This nuisance is avoided by Pletnev and Banin by considering

$$\begin{aligned}&\langle x|f(\mathsf{m},\mathsf{D})|x\rangle \\ &\quad= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\langle x|\exp(-\partial_p\mathsf{D})f(\mathsf{m},\mathsf{D}+p)\exp(\partial_p\mathsf{D})|0\rangle \\ &\quad= \int \frac{\mathrm{d}^d p}{(2\pi)^d}\langle x|f(\bar{\mathsf{m}},\bar{\mathsf{D}})|0\rangle. \qquad (61)\end{aligned}$$

The first equality follows because the momentum derivative $\partial^p_\mu = \partial/\partial p_\mu$ in the last $\exp(\partial_p\mathsf{D})$ factor has no effect since there is no p_μ dependence at its right. Similarly the first factor $\exp(-\partial_p\mathsf{D})$ changes nothing, by integration by parts in the momentum integration. The second equality uses that $\exp(-\partial_p\mathsf{D})\mathsf{X}\exp(\partial_p\mathsf{D})$ defines a similarity transformation. Actually the full similarity transformations is

$$\mathsf{X} \to \bar{\mathsf{X}} = \exp(-\partial_p\mathsf{D})\exp(-xp)\mathsf{X}\exp(xp)\exp(\partial_p\mathsf{D}).$$

The inner transformation produces $\mathsf{m} \to \mathsf{m}$ and $\mathsf{D}_\mu \to \mathsf{D}_\mu + p_\mu$, and is the one used to arrive to the symbols method formula. Now, explicit computation gives [16]

$$\begin{aligned}\bar{\mathsf{m}} &= \mathsf{m} - \hat{\mathsf{D}}_\mu\mathsf{m}\partial^p_\mu + \frac{1}{2!}\hat{\mathsf{D}}_\nu\hat{\mathsf{D}}_\mu\mathsf{m}\partial^p_\nu\partial^p_\mu \\ &\quad - \frac{1}{3!}\hat{\mathsf{D}}_\alpha\hat{\mathsf{D}}_\nu\hat{\mathsf{D}}_\mu\mathsf{m}\partial^p_\alpha\partial^p_\nu\partial^p_\mu + \cdots, \\ \bar{\mathsf{D}}_\mu &= p_\mu - \frac{1}{2!}\mathsf{F}_{\nu\mu}\partial^p_\nu + \frac{2}{3!}\hat{\mathsf{D}}_\alpha\mathsf{F}_{\nu\mu}\partial^p_\alpha\partial^p_\nu \\ &\quad - \frac{3}{4!}\hat{\mathsf{D}}_\beta\hat{\mathsf{D}}_\alpha\mathsf{F}_{\nu\mu}\partial^p_\beta\partial^p_\alpha\partial^p_\nu + \cdots \qquad (62)\end{aligned}$$

As usual $\hat{\mathsf{D}}_\mu\mathsf{X}$ stands for $[\mathsf{D}_\mu,\mathsf{X}]$, the chiral covariant derivative of X. The operator ∂^p_μ denotes the derivative with respect to the p_μ dependence. It acts by taking the derivative of everything to its right (or to its left, by parts). The point of doing this is that the operators ∂_μ (derivative with respect to x_μ) appear only through $\hat{\mathsf{D}}_\mu$ and so

(i) gauge covariance is manifest and
(ii) the integrand is just a function of x (rather than a pseudo-differential operator as $f(\mathsf{m},\mathsf{D})$). This last fact allows one to write

$$\langle x|f(\mathsf{m},\mathsf{D})|x\rangle = \int \frac{\mathrm{d}^d p}{(2\pi)^d}f(\bar{\mathsf{m}},\bar{\mathsf{D}}), \qquad (63)$$

where $f(\bar{\mathsf{m}},\bar{\mathsf{D}})$ is a matrix valued function of x.

In our case the application of this method amounts to replacing (56) by

$$\delta W^-[v,m] = \int \mathrm{d}^d x \mathrm{tr}\left[\gamma_5\delta\not{v}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{\bar{\not{\mathsf{D}}}+\bar{\mathsf{m}}}\right], \quad (64)$$

Note that tr refers to Dirac and flavor spaces here.

The calculation proceeds as follows. The formula (64) is expanded in the number of covariant derivatives, or equivalently in the number of Lorentz indices carried by $\hat{\mathsf{D}}_\mu$ and $\mathsf{F}_{\mu\nu}$. At leading order the term with $d-1$ spatial indices is selected. The derivatives with respect to p_μ are carried out. The Dirac trace is taken. This produces a Levi-Civita pseudo-tensor and differential geometry notation can be used. Note that terms with two or more ∂^p_μ in $\bar{\mathsf{m}}$ and $\bar{\mathsf{D}}_\mu$ cancel, since the corresponding indices are symmetrized. Next, the m are indexed according to Convention 3, thereby becoming c-numbers. This allows one to carry out the momentum integrations straightforwardly; the integration formulas of [1] apply.

A technical detail is that, computationally, the Dirac algebra is slightly alleviated by rewriting (56) as

$$\begin{aligned}&\delta W^-[v,m] \\ &\quad= \int \mathrm{d}^d x\mathrm{tr}\left[\langle x|\gamma_5\delta\not{v}\frac{1}{(\not{\mathsf{D}}-\mathsf{m})(\not{\mathsf{D}}+\mathsf{m})}(\not{\mathsf{D}}-\mathsf{m})|x\rangle\right] \\ &\quad= -\int \mathrm{d}^d x\mathrm{tr}\Bigg[\langle x|\gamma_5\delta\not{v}\frac{1}{-\mathsf{D}_\mu^2+\mathsf{m}^2-\hat{\not{\mathsf{D}}}\mathsf{m}-\frac{1}{2}\sigma_{\mu\nu}\mathsf{F}_{\mu\nu}} \\ &\qquad\times(\not{\mathsf{D}}-\mathsf{m})|x\rangle\Bigg]. \qquad (65)\end{aligned}$$

The formulas in (62) define a similarity transformation [16], so the replacements $\mathsf{m}\to\bar{\mathsf{m}}$ and $\mathsf{D}_\mu\to\bar{\mathsf{D}}_\mu$ apply here too.

Let us illustrate this procedure for the two-dimensional case. Applying the replacements $\mathsf{m}\to\bar{\mathsf{m}}$ and $\mathsf{D}_\mu\to\bar{\mathsf{D}}_\mu$ in the second of (65), and retaining terms with at most one covariant derivative, yields

$$\begin{aligned}&\delta W^-_{d=2}[v,m] \\ &= -\int\frac{\mathrm{d}^2x\mathrm{d}^2p}{(2\pi)^2}\mathrm{tr}\left[\gamma_5\delta\not{v}\frac{1}{\varDelta-\hat{\not{\mathsf{D}}}\mathsf{m}-\{\mathsf{m},\hat{\mathsf{D}}_\mu\mathsf{m}\}\partial^p_\mu}(\not{p}-\mathsf{m})\right] \\ &= -\int\frac{\mathrm{d}^2x\mathrm{d}^2p}{(2\pi)^2}\mathrm{tr}\Bigg[\gamma_5\delta\not{v}\frac{1}{\varDelta}\left(\hat{\not{\mathsf{D}}}\mathsf{m}+\{\mathsf{m},\hat{\mathsf{D}}_\mu\mathsf{m}\}\partial^p_\mu\right) \\ &\quad\times\frac{1}{\varDelta}(\not{p}-\mathsf{m})\Bigg], \qquad (66)\end{aligned}$$

where we have defined $\varDelta = p^2+\mathsf{m}^2$ (not to be confused with the operator $\varDelta$ introduced in (19)). The formula is

[6] Our notation will be as follows: p_μ is purely imaginary; however, $\int\mathrm{d}^d p$ denotes the standard integration on $\mathbf{R}^d$ and p^2 denotes $-p_\mu p_\mu$.

already ultraviolet convergent without further renormalization. This was to be expected since the chiral covariant current is unique and thus free from ultraviolet ambiguities. Using the formulas

$$\partial^p_\mu \frac{1}{\Delta} = \frac{2p_\mu}{\Delta^2}, \quad \gamma_5\gamma_\mu\gamma_\nu \to -\eta_2\epsilon_{\mu\nu}, \quad p_\mu p_\nu \to -\frac{p^2}{2}\delta_{\mu\nu}, \tag{67}$$

the expression becomes

$$\begin{aligned}\delta W^-_{d=2}[v,m] &= -2\eta_2 \int \frac{\mathrm{d}^2p}{(2\pi)^2} \mathrm{tr}\left[\delta\mathsf{v}\left(\frac{1}{\Delta}\mathsf{m}'\frac{\mathsf{m}}{\Delta} - p^2\frac{1}{\Delta^2}\{\mathsf{m},\mathsf{m}'\}\frac{1}{\Delta}\right)\right] \\ &= 2\eta_2 \int \frac{\mathrm{d}^2p}{(2\pi)^2} \mathrm{tr}\left[\left(\frac{\mathsf{m}_2}{\Delta_1\Delta_2} - p^2\frac{\mathsf{m}_1+\mathsf{m}_2}{\Delta_1^2\Delta_2}\right)\mathsf{m}'\delta\mathsf{v}\right],\end{aligned} \tag{68}$$

where we are already using the notation of differential forms, and $\mathsf{m}' = \hat{\mathsf{D}}\mathsf{m}$. The trace no longer includes Dirac space. The integration over momenta can be done using the formulas in [1] and the result in (59) follows.

The calculation in four dimensions is similar and yields

$$\begin{aligned}\delta W^-_{d=4}[v,m] &= 4\eta_4 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{tr}\left[\left(\frac{\mathsf{m}_3}{\Delta_1\Delta_2\Delta_3\Delta_4} + \frac{p^2}{2}\left(\frac{\mathsf{m}_1-\mathsf{m}_3}{\Delta_1^2\Delta_2\Delta_3\Delta_4}\right.\right.\right. \\ &\quad \left.\left. - \frac{\mathsf{m}_2+\mathsf{m}_3}{\Delta_1\Delta_2^2\Delta_3\Delta_4} - \frac{\mathsf{m}_3+\mathsf{m}_4}{\Delta_1\Delta_2\Delta_3\Delta_4^2}\right)\right)\mathsf{m}'^3\delta\mathsf{v} \\ &\quad + \left(\frac{\mathsf{m}_1}{\Delta_1\Delta_2\Delta_3} - \frac{p^2}{2}\left(\frac{\mathsf{m}_1-\mathsf{m}_2}{\Delta_1\Delta_2^2\Delta_3} + \frac{\mathsf{m}_1+\mathsf{m}_3}{\Delta_1\Delta_2\Delta_3^2}\right)\right)\mathsf{F}\mathsf{m}'\delta\mathsf{v} \\ &\quad \left. + \left(\frac{\mathsf{m}_1}{\Delta_1\Delta_2\Delta_3} - \frac{p^2}{2}\left(\frac{\mathsf{m}_1+\mathsf{m}_2}{\Delta_1\Delta_2^2\Delta_3} + \frac{\mathsf{m}_1+\mathsf{m}_3}{\Delta_1\Delta_2\Delta_3^2}\right)\right)\mathsf{m}'\mathsf{F}\delta\mathsf{v}\right].\end{aligned} \tag{69}$$

Integration over momentum yields the results quoted in (B3). It can be noted that the integrands in (68) and (69) are not unique, due to integration by parts in momentum space. On the other hand, their integrals, the functions A, are unambiguous.

4 The effective action

Following the strategy outlined above, we should now consider the most (or, at least, a sufficiently) general effective action functional in the pseudo-parity odd sector and at leading order in the covariant derivative expansion, consistent with Lorentz and chiral symmetries. This will be done by writing the effective action as

$$W^-[v,m] = \Gamma_{\mathrm{gWZW}}[v,m] + W^-_{\mathrm{c}}[v,m]. \tag{70}$$

The functional $\Gamma_{\mathrm{gWZW}}[v,m]$, an extended gauged Wess–Zumino–Witten (gWZW) action, is chosen in order to reproduce the correct chiral anomaly. The extension refers to the fact that it goes beyond the chiral circle constraint. Once the anomaly is saturated, the remainder will be chiral invariant and can be adjusted in order to reproduce the known current. This chiral invariant remainder is denoted by $W^-_{\mathrm{c}}[v,m]$.

4.1 The extended gauged Wess-Zumino-Witten action

As is well known (see Appendix A), the ordinary gauged WZW functional $\Gamma_{\mathrm{LR}}[v_{\mathrm{L}},v_{\mathrm{R}},U]$ reproduces the correct chiral anomaly (in the LR version). Two essential properties of this result are
(i) that it follows solely from assuming the transformation property $U \to \Omega_{\mathrm{L}}^{-1}U\Omega_{\mathrm{R}}$, and no other algebraic properties on $U(x)$, and
(ii) the infinitesimal chiral variation of $\Gamma_{\mathrm{LR}}[v_{\mathrm{L}},v_{\mathrm{R}},U]$ (i.e. the anomaly) depends on the gauge fields $v_{\mathrm{L,R}}$ but not on U. In view of this, we can use m_{LR} instead of U in order to reproduce the anomaly. The antisymmetry under pseudo-parity conjugation can be reestablished using the fact that this conjugation commutes with chiral transformations. Therefore, the following functional serves as the extended gauged Wess–Zumino–Witten (gWZW) action:

$$\Gamma_{\mathrm{gWZW}}[v,m] = \frac{1}{2}\Gamma_{\mathrm{LR}}[v_{\mathrm{L}},v_{\mathrm{R}},m_{\mathrm{LR}}] - \frac{1}{2}\Gamma_{\mathrm{LR}}[v_{\mathrm{R}},v_{\mathrm{L}},m_{\mathrm{RL}}]. \tag{71}$$

In this functional the two fields m_{LR} and m_{RL} are not mixed. This is not a property of the full effective action, as is already clear from the form of the effective current computed in the previous section.

In order to write this functional using our conventions, let us consider the contribution of the (ungauged) WZW term in two dimensions (cf. (A13) setting v to zero); we have

$$\Gamma_{\mathrm{WZW},d=2}[m] = -\frac{1}{6}\left\langle\left(\frac{1}{m_{\mathrm{LR}}}\mathrm{d}m_{\mathrm{LR}}\right)^3 - \left(\frac{1}{m_{\mathrm{RL}}}\mathrm{d}m_{\mathrm{RL}}\right)^3\right\rangle. \tag{72}$$

This can be rewritten as

$$\Gamma_{\mathrm{WZW},d=2}[m] = \left\langle -\frac{1}{3}\mathsf{R}^3\right\rangle. \tag{73}$$

The meaning of the symbol $\langle\ \rangle$ was given in (3). We have introduced the 1-form $\mathsf{R} = (1/\mathsf{m})\mathrm{d}\mathsf{m}$, and Conventions 1 and 2 apply. Note that, consistently with $\mathsf{m}^{-1}\mathsf{m} = 1$, $(\mathsf{m}^{-1})_{\mathrm{LR}} = m^{-1}_{\mathrm{RL}}$ and $(\mathsf{m}^{-1})_{\mathrm{RL}} = m^{-1}_{\mathrm{LR}}$. More generally, in d dimensions

$$\Gamma_{\mathrm{WZW}}[m] = \left\langle -\frac{1}{d+1}\mathsf{R}^{d+1}\right\rangle. \tag{74}$$

As usual, in $\Gamma_{\mathrm{WZW}}[m]$ the integration takes place on a $d+1$-dimensional disk with the d-dimensional spacetime as boundary. It is essential that the integrand is a closed form, so that the result does not depend on topologically small deformations of the $d+1$-dimensional disk. This property follows from $\mathrm{d}\mathsf{R} = -\mathsf{R}^2$ and the cyclic property. On the other hand the normalization is such that $\Gamma_{\mathrm{WZW}}[m]$ changes by integer multiples of $2\pi\mathrm{i}$ under large deformations of the disk; this holds when m is on the chiral circle and the difference between $\mathrm{tr}(\mathsf{R}^{d+1})$ on and off the chiral circle is an exact form. (We are assuming throughout that the fields $m_{\mathrm{LR}}(x)$ and $m_{\mathrm{RL}}(x)$ are nowhere singular, so any configuration can be deformed to one on the chiral circle.)

The full gauged functional in zero, two and four dimensions takes the form

$$\begin{aligned}
&\Gamma_{\mathrm{gWZW},d=0}[v,m] = \langle -\mathsf{R}_c - 2\mathsf{v}\rangle = \langle -\mathsf{R}\rangle\,, \\
&\Gamma_{\mathrm{gWZW},d=2}[v,m] \\
&= \left\langle -\frac{1}{3}\mathsf{R}_c^3 + (\mathsf{R}_c + \mathsf{L}_c)\mathsf{F} + 2\mathsf{vF} - \frac{2}{3}\mathsf{v}^3 \right\rangle \\
&= \left\langle -\frac{1}{3}\mathsf{R}^3 - (\mathsf{R}+\mathsf{L})\mathsf{v} - \mathsf{mvm}^{-1}\mathsf{v} \right\rangle, \qquad (75) \\
&\Gamma_{\mathrm{gWZW},d=4}[v,m] = \left\langle -\frac{1}{5}\mathsf{R}_c^5 + (\mathsf{R}_c^3 + \mathsf{L}_c^3)\mathsf{F} \right. \\
&-2(\mathsf{R}_c + \mathsf{L}_c)\mathsf{F}^2 - \mathsf{R}_c\mathsf{Fm}^{-1}\mathsf{Fm} - \mathsf{L}_c\mathsf{FmFm}^{-1} \\
&\left. -4\mathsf{vF}^2 + 2\mathsf{v}^3\mathsf{F} - \frac{2}{5}\mathsf{v}^5 \right\rangle \\
&= \left\langle -\frac{1}{5}\mathsf{R}^5 - (\mathsf{R}^3 + \mathsf{L}^3)\mathsf{v} + \frac{1}{2}(\mathsf{Rv})^2 + \frac{1}{2}(\mathsf{Lv})^2 \right. \\
&+\mathsf{R}^2\mathsf{vm}^{-1}\mathsf{vm} + \mathsf{L}^2\mathsf{vmvm}^{-1} \\
&+\mathsf{Rm}^{-1}\mathsf{vmdv} + \mathsf{Lmvm}^{-1}\mathsf{dv} \\
&+(\mathsf{R}+\mathsf{L})\mathsf{v}^3 + \mathsf{Rvm}^{-1}\mathsf{vmv} + \mathsf{Lvmvm}^{-1}\mathsf{v} \\
&+(\mathsf{R}+\mathsf{L}+\mathsf{m}^{-1}\mathsf{vm} + \mathsf{mvm}^{-1})\{\mathsf{v},\mathsf{dv}\} \\
&\left. +\mathsf{mvm}^{-1}\mathsf{v}^3 + \mathsf{m}^{-1}\mathsf{vmv}^3 + \frac{1}{2}(\mathsf{mvm}^{-1}\mathsf{v})^2 \right\rangle.
\end{aligned}$$

The functional $\Gamma_{\mathrm{gWZW}}[v,m]$ corresponds to the LR version of the action (as opposed to the VA version). In these formulas we have introduced the following 1-forms:

$$\begin{aligned}
\mathsf{R} &= \mathsf{m}^{-1}\mathsf{dm} = -\mathsf{dm}^{-1}\mathsf{m}, \\
\mathsf{L} &= \mathsf{mdm}^{-1} = -\mathsf{dmm}^{-1} = -\mathsf{mRm}^{-1}, \qquad (76) \\
\mathsf{R}_c &= \mathsf{m}^{-1}\hat{\mathsf{D}}\mathsf{m} = \mathsf{R} + \mathsf{m}^{-1}\mathsf{vm} - \mathsf{v} = \mathsf{m}^{-1}\mathsf{m}', \\
\mathsf{L}_c &= \mathsf{m}\hat{\mathsf{D}}\mathsf{m}^{-1} = -\hat{\mathsf{D}}\mathsf{mm}^{-1} = \mathsf{L} + \mathsf{mvm}^{-1} - \mathsf{v} \\
&= -\mathsf{mR}_c\mathsf{m}^{-1} = -\mathsf{m}'\mathsf{m}^{-1}.
\end{aligned}$$

R_c and L_c are covariant under chiral gauge transformations.

The functional $\Gamma_{\mathrm{gWZW}}[v,m]$ has been written in two different forms in (75). In the first version all terms in the integrand are $d+1$-forms. This version shows explicitly that the pieces which break chiral symmetry can be written as an m-independent polynomial (in fact, this polynomial is just the correctly normalized Chern–Simons term in $d+1$ dimensions). This guarantees that the corresponding chiral anomaly will also be an m-independent polynomial. Technically, such a piece looks like an ordinary counterterm which could be removed from the effective action, leaving a chiral invariant action. Of course, this procedure would be incorrect, since this piece, as well as the remainder, is not a separately closed form. Note that the $d+1$ component of v does not really appear in the functional (first version) since it cancels identically (this is easily seen in the 0-dimensional case). In the second version, all contributions in the integrand, except for the WZW term, are d-forms. In this version the chiral symmetry is less obvious, but it is closer to an ordinary d-dimensional Lagrangian.

Under the mirror transformation introduced in the previous section, the terms which are d-forms are odd, whereas those written as $d+1$-forms are even. Using $\mathsf{R}\to\mathsf{L}$, $\mathsf{R}_c\to\mathsf{L}_c$, etc., one has, for instance,

$$\begin{aligned}
&\langle \mathsf{R}_c\mathsf{Fm}^{-1}\mathsf{Fm}\rangle \to \langle \mathsf{mFm}^{-1}\mathsf{FL}_c\rangle = \langle \mathsf{L}_c\mathsf{mFm}^{-1}\mathsf{F}\rangle \\
&= -\langle \mathsf{mR}_c\mathsf{Fm}^{-1}\mathsf{F}\rangle = +\langle \mathsf{R}_c\mathsf{Fm}^{-1}\mathsf{Fm}\rangle. \qquad (77)
\end{aligned}$$

To finish this section, let us comment on the possibility of writing the functional $\Gamma_{\mathrm{gWZW}}[v,m]$ as a d-form. This has already been done for the gauged terms. The question is whether the WZW term $\Gamma_{\mathrm{WZW}}[m]$, (74), can also be written as a d-form in terms of m. In Sect. 2.2 this was done for $\Gamma_{\mathrm{WZW}}[U]$. The same formula does not directly apply, because m behaves differently under the cyclic property, namely, it is odd, whereas U is even. Indeed, if we try to use the same method, we can see that it fails, since any deformation of m into some $\mathsf{m}(t)$ with $\mathsf{m}(0)=1$ is in conflict with the condition that the expressions must be even functions of m, for consistency. A possibility is to go back to (71) (with $v=0$) and observe that $\Gamma_{\mathrm{WZW}}[m_{\mathrm{LR}}]$ and $\Gamma_{\mathrm{WZW}}[m_{\mathrm{RL}}]$ are invariant under the replacements $m_{\mathrm{LR}} \to M_{\mathrm{LR}}^{-1}m_{\mathrm{LR}}$ and $m_{\mathrm{RL}} \to M_{\mathrm{RL}}^{-1}m_{\mathrm{RL}}$, where M_{LR} and M_{RL} are spacetime constants. Defining the new symbol M by $(\mathsf{M})_{\mathrm{LR}} = M_{\mathrm{LR}}$ and $(\mathsf{M})_{\mathrm{RL}} = M_{\mathrm{RL}}$, the WZW term can be written as

$$\Gamma_{\mathrm{WZW}}[m] = \Gamma_{\mathrm{WZW}}[U], \quad U = \mathsf{M}^{-1}\mathsf{m}. \qquad (78)$$

m and M are odd under the cyclic property, whereas $\mathsf{M}^{-1}\mathsf{m}$ is even; therefore the formulas derived in Section 2.2 apply directly.

4.2 The chiral invariant remainder

Using integration by parts, the most general functionals consistent with chiral invariance and Lorentz invariance (and at leading order in the derivative expansion) are of the form[7]

$$\begin{aligned}
W^-_{c,d=2}[v,m] &= \left\langle N(\mathsf{m}_1,\mathsf{m}_2)(\hat{\mathsf{D}}\mathsf{m})^2 \right\rangle := \langle N_{12}\mathsf{m}'^2\rangle\,, \\
W^-_{c,d=4}[v,m] &= \langle N_{1234}\mathsf{m}'^4 + N_{123}\mathsf{m}'^2\mathsf{F}\rangle\,. \qquad (79)
\end{aligned}$$

A comment is in order here. We have already argued above that the current uniquely determines the effective action. Therefore, it is not strictly necessary to deal with the most general class of chiral invariant functionals. Any class of functionals can be used, provided that it happens to contain the correct $W^-_c[v,m]$. The reason why the form in (79) is sufficiently general is not entirely straightforward, since one could imagine terms of the form $\langle N_1\mathsf{F}\rangle$ in two dimensions, or $\langle N_{12}\mathsf{F}^2\rangle$ in four dimensions. In Sect. 5.6 we

[7] $W^-_c[v,m]$ vanishes in 0 dimensions. An easy calculation shows that (choosing $\eta_0 = 1$) $W[v,m] = -\log(m_{\mathrm{LR}})$ and thus $\Gamma_{\mathrm{gWZW}}[v,m]$ is the full result in this case. In addition, the most general form of W^-_c would be $\langle f(\mathsf{m}^2)\rangle$, but f has to be a constant due to scale invariance, and hence it vanishes in the pseudo-parity odd sector.

show that those terms are in fact redundant. On the other hand, far more general chiral covariant terms can be devised. In (79) we have imposed that the functional must be an analytical function of m and D. More general functionals exist if this condition is lifted. However, the analytical form is sufficient for the effective action functional. Note that the analytical form comes out automatically for the effective current, as a result of the calculation. These more general chiral covariant functionals are discussed in Sect. 5.6.

Let us discuss which restrictions exist on the functions N in $W_{\rm c}^-[v,m]$. The cyclic property implies

$$N_{12}=N_{2\underline{1}},\quad N_{1234}=N_{234\underline{1}}. \tag{80}$$

(That is, $N(\mathsf{m}_1,\mathsf{m}_2)=N(\mathsf{m}_2,-\mathsf{m}_1)$, etc.) This already implies "consistency" (i.e., the functions N should be even under $\mathsf{m}\to-\mathsf{m}$). We have as a byproduct

$$N_{12}=N_{\underline{12}},\quad N_{1234}=N_{\underline{1234}},\quad N_{123}=N_{\underline{123}}. \tag{81}$$

Mirror symmetry requires

$$N_{12}=-N_{21},\quad N_{1234}=-N_{4321},\quad N_{123}=-N_{321}. \tag{82}$$

Note the different nature of the constraints implied by the cyclic property and mirror symmetry. Mirror symmetry is a property of our particular functional $W_{\rm c}^-[v,m]$, and it is perfectly possible to write non-null terms violating this symmetry. On the other hand, the cyclic symmetry is automatic; any function N_{12} can be decomposed under the group generated by $12\to2\underline{1}$, and only that component satisfying $N_{12}=N_{2\underline{1}}$ can have a non-vanishing contribution to the functional. Thus (80) expresses our choice of working with this relevant component only.

Dimensional counting implies that N_{12} and N_{123} have dimensions of $[m^{-2}]$, and N_{1234} of $[m^{-4}]$. In addition, the functions N must be regular as two or more arguments coincide up to a sign.

It is important to note that the functions N in four dimensions are not unambiguously determined by the functional itself, due to integration by parts. This follows from the identity

$$\begin{aligned}0&=-\frac{1}{3}\left\langle\hat{\mathsf{D}}\left(H_{123}\mathsf{m}'^3\right)\right\rangle\\&=\left\langle-\frac{1}{3}(\Delta H)_{1234}\mathsf{m}'^4+(\mathsf{m}_1+\mathsf{m}_3)H_{123}\mathsf{m}'^2\mathsf{F}\right\rangle.\end{aligned} \tag{83}$$

The operator Δ was defined in (19) and the identity $\hat{\mathsf{D}}^2\mathsf{m}=[\mathsf{F},\mathsf{m}]$ has been used. In addition, the cyclic property has been assumed on H_{123}. For subsequent reference we give the cyclic property and mirror symmetry conditions on H_{123}:

$$H_{123}=-H_{\underline{123}}=-H_{23\underline{1}}=-H_{321}. \tag{84}$$

The identity in (83) implies that there is an ambiguity in the definition of N_{123}, since it can always be augmented by $(\mathsf{m}_1+\mathsf{m}_3)H_{123}$ with arbitrary H_{123} subjected to the conditions just quoted, and similarly for N_{1234}. (Note that $(\Delta H)_{1234}$ does not directly have the cyclic property assumed for N_{1234} – it has to be symmetrized.)

4.3 Results

First, we will need to compute the contribution of $\Gamma_{\rm gWZW}[v,m]$ to the effective current. In two dimensions, a first order variation with respect to v yields

$$\delta\Gamma_{{\rm gWZW},d=2}[v,m]=\langle(-\mathsf{R}_{\rm c}-\mathsf{L}_{\rm c}-2\mathsf{v})\delta\mathsf{v}\rangle\quad(\delta\mathsf{m}=0). \tag{85}$$

Two contributions to the consistent current are identified which correspond to the covariant contribution and the counterterm in (35):

$$\mathsf{J}^{\rm WZW}_{v,c,d=2}=-\mathsf{R}_{\rm c}-\mathsf{L}_{\rm c},\quad\mathsf{P}_{d=2}(v)=-2\mathsf{v}. \tag{86}$$

Similarly, in four dimensions

$$\begin{aligned}\mathsf{J}^{\rm WZW}_{v,c,d=4}&=-\mathsf{R}_{\rm c}^3-\mathsf{L}_{\rm c}^3+2\{\mathsf{R}_{\rm c}+\mathsf{L}_{\rm c},\mathsf{F}\}-\mathsf{m}\{\mathsf{R}_{\rm c},\mathsf{F}\}\mathsf{m}^{-1}\\&\quad-\mathsf{m}^{-1}\{\mathsf{L}_{\rm c},\mathsf{F}\}\mathsf{m},\\\mathsf{P}_{d=4}(v)&=4\{\mathsf{v},\mathsf{F}\}-2\mathsf{v}^3.\end{aligned} \tag{87}$$

(The same polynomials $\mathsf{P}(v)$ would be obtained with any choice of $\Gamma_{\rm gWZW}[v,m]$, since they are completely fixed by the chiral anomaly.)

The contribution of $W_{\rm c}^-[v,m]$ to the current is of course purely covariant and it can be read off from

$$\begin{aligned}&\delta W^-_{c,d=2}[v,m]=\langle-2(\mathsf{m}_1+\mathsf{m}_2)N_{12}\mathsf{m}'\delta\mathsf{v}\rangle\,,\\&\delta W^-_{c,d=4}[v,m]\\&\quad=\big\langle\big(-(\Delta N)_{1234}-4(\mathsf{m}_1+\mathsf{m}_4)N_{1234}\big)\,\mathsf{m}'^3\delta\mathsf{v}\\&\quad+\big((\mathsf{m}_1-\mathsf{m}_2)N_{123}-(\mathsf{m}_1+\mathsf{m}_3)N_{23\underline{1}}\big)\,\mathsf{F}\mathsf{m}'\delta\mathsf{v}\\&\quad+\big((\mathsf{m}_2-\mathsf{m}_3)N_{321}+(\mathsf{m}_1+\mathsf{m}_3)N_{21\underline{3}}\big)\,\mathsf{m}'\mathsf{F}\delta\mathsf{v}\big\rangle\,.\end{aligned} \tag{88}$$

In these formulas, the cyclic property has explicitly been assumed for the functions N_{12} and N_{1234}. The operator Δ was defined in (19):

$$(\Delta N)_{1234}=\frac{N_{134}-N_{234}}{\mathsf{m}_1-\mathsf{m}_2}-\frac{N_{124}-N_{134}}{\mathsf{m}_2-\mathsf{m}_3}+\frac{N_{123}-N_{124}}{\mathsf{m}_3-\mathsf{m}_4}. \tag{89}$$

Collecting the different contributions to the covariant current in (86), (87) and (88), and comparing with the definition of the functions A in (36), the following relations are derived:

$$\begin{aligned}A_{12}&=-\frac{1}{\mathsf{m}_1}+\frac{1}{\mathsf{m}_2}-2(\mathsf{m}_1+\mathsf{m}_2)N_{12},\\A_{123}&=\frac{1}{\mathsf{m}_1}+\frac{2}{\mathsf{m}_2}-\frac{2}{\mathsf{m}_3}-\frac{\mathsf{m}_1}{\mathsf{m}_2\mathsf{m}_3}+(\mathsf{m}_1-\mathsf{m}_2)N_{123}\\&\quad-(\mathsf{m}_1+\mathsf{m}_3)N_{23\underline{1}},\\A_{1234}&=-\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3}+\frac{1}{\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4}-(\Delta N)_{1234}\\&\quad-4(\mathsf{m}_1+\mathsf{m}_4)N_{1234}.\end{aligned} \tag{90}$$

The terms containing N are those coming from $W_{\rm c}^-[v,m]$, whereas the explicit terms are those coming from $\Gamma_{\rm gWZW}[v,m]$. In these relations the N's are the unknown. It is important to note that these relations have to be augmented with the cyclicity constraints, (80), since they have explicitly been used in their derivation.

Let us consider the two-dimensional case. For N_{12} one obtains

$$N_{12} = -\frac{1}{2}\frac{1}{\mathsf{m}_1+\mathsf{m}_2}\left(A_{12}+\frac{1}{\mathsf{m}_1}-\frac{1}{\mathsf{m}_2}\right) \tag{91}$$

$$= -\frac{\mathsf{m}_1\mathsf{m}_2}{\mathsf{m}_1^2-\mathsf{m}_2^2}\left(\frac{\log(\mathsf{m}_1^2/\mathsf{m}_2^2)}{\mathsf{m}_1^2-\mathsf{m}_2^2}-\frac{1}{2}\left(\frac{1}{\mathsf{m}_1^2}+\frac{1}{\mathsf{m}_2^2}\right)\right). \tag{92}$$

It is worth noticing that the correct cyclic property for N_{12}, namely, $N_{12} = N_{2\underline{1}}$, is verified, but this is not an automatic consequence of (91). This poses a severe restriction on the a priori admissible functions A_{12}, if they should derive from an effective action with the correct Lorentz and chiral symmetries. In addition the function N_{12} is finite, i.e. regular at $\mathsf{m}_1^2 = \mathsf{m}_2^2$. Again, this property does not follow automatically from the finiteness of A_{12}. On the other hand, scale and mirror symmetries are automatic in N_{12} from the corresponding symmetries in A_{12}.

Another comment is that in the first of (90) (similar remarks apply to the four-dimensional formulas) the WZW contribution to A_{12} cannot be reabsorbed into the contribution coming from $W_{\mathrm{c}}^-[v,m]$ by means of a suitable redefinition of the function N_{12}. If this were the case, we would have that the covariant current, A_{12}, is also consistent (it would derive from a certain $W_{\mathrm{c}}^-[v,m]$). Technically, the reason is that such a function N_{12} would violate the cyclic property constraint. (In addition, it would not be finite at $\mathsf{m}_2 = -\mathsf{m}_1$.) Thus, in this formalism the non-integrability of the covariant current, which necessarily implies the existence of a chiral anomaly, translates into a breakdown of the cyclic property.

In summary, the function N_{12} in (92) inserted in W_{c}^- $[v,m]$ in (79) plus the extended gauged WZW term in (75) provides the full functional for the leading order of the pseudo-parity odd component of the effective action in two dimensions.

Let us now turn to the four-dimensional case. Unfortunately, this case is more involved, mainly because of the presence of ambiguities in the functions N introduced by integration by parts. These ambiguities do not affect the functional $W_{\mathrm{c}}^-[v,m]$ itself.

Mirror symmetry of N_{123} automatically implies $\bar{A}_{123} = \bar{A}_{132}$, which can thus be understood as a consequence of mirror plus Lorentz symmetries (chiral symmetry is not required). The full permutation symmetry of $\bar{A}_{123}$ and $\bar{A}_{1234}$ would not follow if $N_{123} = N_{1234} = 0$ and so it cannot be understood in this way.

Clearly, N_{1234} is uniquely determined by the formulas once A_{1234} and N_{123} are known. However, N_{123} is not unambiguously determined from A_{123}. In turn, this puts a restriction on the possible A_{123}, namely,

$$(\mathsf{m}_2-\mathsf{m}_3)A_{123}+(\mathsf{m}_1+\mathsf{m}_3)A_{23\underline{1}}+(\mathsf{m}_2-\mathsf{m}_1)A_{3\underline{12}} = 12, \tag{93}$$

which is verified by the true function A_{123}. This ambiguity was noted above, see (83). It is verified that the modification introduced by H_{123} exactly cancels in the right-hand side of (90). This serves as a check of these formulas. This means that the functions N_{123} and N_{1234} are ambiguous, but not the functional $W_{\mathrm{c}}^-[v,m]$ itself. This is consistent with the fact that the current completely fixes the effective action functional if the correct chiral transformation is assumed.

A particular solution for the functional N_{123} is given by

$$N_{123}^0 = \frac{1}{3}\left(\frac{A_{123}}{\mathsf{m}_1-\mathsf{m}_2}+\frac{A_{3\underline{12}}}{\mathsf{m}_2-\mathsf{m}_3}\right) - \frac{1}{(\mathsf{m}_1-\mathsf{m}_2)(\mathsf{m}_2-\mathsf{m}_3)}\left(\frac{\mathsf{m}_1}{\mathsf{m}_2}+\frac{\mathsf{m}_2}{\mathsf{m}_1}-\frac{\mathsf{m}_2}{\mathsf{m}_3}-\frac{\mathsf{m}_3}{\mathsf{m}_2}\right). \tag{94}$$

This is easily verified by substitution. The associated function N_{1234}^0 is immediately obtained from (90). Besides the trivial mirror symmetry and scale invariances, it is verified that N_{1234}^0 possesses the correct cyclic symmetry. Again this is a highly non-trivial check of the functions A_{123} and A_{1234}. However, the functions N_{123}^0 and N_{1234}^0 are not directly acceptable, since they fail to be finite in the coincidence limit, namely, when $\mathsf{m}_1 = \mathsf{m}_2$ or $\mathsf{m}_2 = \mathsf{m}_3$. This implies that another solution has to be chosen by taking an appropriate function H_{123}. (Note that the previous checks are preserved by this operation.)

To find an acceptable solution it is convenient to work with a reduced version of the function N_{123}, namely

$$\widehat{N}_{123} = (\mathsf{m}_1-\mathsf{m}_2)(\mathsf{m}_2-\mathsf{m}_3)N_{123}. \tag{95}$$

Consistency and mirror symmetry of N_{123} translate into

$$\widehat{N}_{123} = \widehat{N}_{\underline{123}} = -\widehat{N}_{321}. \tag{96}$$

On the other hand, the condition of finiteness of N_{123} at $\mathsf{m}_1 = \mathsf{m}_2$ corresponds to

$$\widehat{N}_{113} = 0. \tag{97}$$

Due to mirror symmetry this immediately implies $\widehat{N}_{122} = 0$ and, thus, finiteness of N_{123} at $\mathsf{m}_2 = \mathsf{m}_3$ too. This finiteness condition is violated by $\widehat{N}_{123}^0$.

Likewise, for the function H_{123} controlling the ambiguity we define its reduced version by

$$\widehat{H}_{123} = (\mathsf{m}_1-\mathsf{m}_2)(\mathsf{m}_2-\mathsf{m}_3)(\mathsf{m}_1+\mathsf{m}_3)H_{123}. \tag{98}$$

Consistency, the cyclic property and mirror symmetry of H_{123} in (84) translate into

$$\widehat{H}_{123} = \widehat{H}_{\underline{123}} = \widehat{H}_{23\underline{1}} = -\widehat{H}_{321}. \tag{99}$$

(This is equivalent to saying that the function $\widehat{H}_{123}$ is completely antisymmetric under permutation of its arguments and even under $\mathsf{m}\to-\mathsf{m}$.)

In terms of the reduced functions, the ambiguity corresponds to the fact that $\widehat{N}_{123}$ and $\widehat{N}_{123}-\widehat{H}_{123}$ produce the same current A_{123}. In view of this, our strategy is to find an $\widehat{H}_{123}$ such that $\widehat{H}_{113} = \widehat{N}_{113}^0$, so that

$$\widehat{N}_{123} = \widehat{N}_{123}^0 - \widehat{H}_{123} \tag{100}$$

fulfills the finiteness condition, (97). This can be done as follows. Although the function $\widehat{N}^0_{123}$ does not vanish at $\mathsf{m}_1 = \mathsf{m}_2$, it is finite and satisfies

$$\widehat{N}^0_{123} = \widehat{N}^0_{\underline{123}}, \quad \widehat{N}^0_{111} = 0, \quad \widehat{N}^0_{113} = -\widehat{N}^0_{11\underline{3}}. \tag{101}$$

The first relation is consistency. The second one comes from mirror symmetry, and the last one follows from finiteness of A_{123} (this is more simply verified from (90)). These relations imply that the function

$$\widehat{H}_{123} = \frac{1}{2}\left(-\frac{\mathsf{m}_3}{\mathsf{m}_1}\widehat{N}^0_{112} + \frac{\mathsf{m}_1}{\mathsf{m}_2}\widehat{N}^0_{223} - \frac{\mathsf{m}_2}{\mathsf{m}_3}\widehat{N}^0_{331} - \frac{\mathsf{m}_3}{\mathsf{m}_2}\widehat{N}^0_{221} + \frac{\mathsf{m}_1}{\mathsf{m}_3}\widehat{N}^0_{332} + \frac{\mathsf{m}_2}{\mathsf{m}_1}\widehat{N}^0_{113}\right) \tag{102}$$

satisfies the requirements in (99). In addition,

$$\widehat{H}_{113} = \frac{1}{2}\left(-\frac{\mathsf{m}_3}{\mathsf{m}_1}\widehat{N}^0_{111} + \widehat{N}^0_{113} - \frac{\mathsf{m}_1}{\mathsf{m}_3}\widehat{N}^0_{331} - \frac{\mathsf{m}_3}{\mathsf{m}_1}\widehat{N}^0_{111} + \frac{\mathsf{m}_1}{\mathsf{m}_3}\widehat{N}^0_{331} + \widehat{N}^0_{113}\right) = \widehat{N}^0_{113}; \tag{103}$$

therefore, the function $\widehat{N}_{123}$ defined as $\widehat{N}^0_{123} - \widehat{H}_{123}$ automatically satisfies $\widehat{N}_{113} = 0$ and thus it yields a finite N_{123}. Finiteness of the corresponding N_{1234} also follows automatically: because N_{123} and A_{1234} are finite, (90) implies that the N_{1234} is also finite except perhaps at $\mathsf{m}_1 = -\mathsf{m}_4$; however, this follows from the cyclic property, $N_{1234} = N_{\underline{4}123}$ and the finiteness of N_{1123}.

Let us summarize the result. The acceptable N_{123} is obtained as follows: from A_{123} (B3), one obtains N^0_{123} (94), then $\widehat{N}^0_{123}$ (95) and $\widehat{H}_{123}$ (102). This gives $\widehat{N}_{123}$ (100) and N_{123} (95). Finally, N_{1234} follows from (90). The explicit resulting functions are displayed in Appendix B.

It can be noted that in addition to the redefinition from N^0_{123} to N_{123} to achieve finiteness, further redefinitions, by suitable finite functions H_{123}, can be made to simplify the final form of N_{123} and N_{1234}. In practice, we have not been able to achieve a greater simplification. Certainly the functions N_{123} and N_{1234} cannot be much simpler than the functions A_{123} and A_{1234}, which are free from ambiguities; thus, no simple form is to be expected for the functions N.

As we have seen in Sect. 2.2, the WZW term has a simple form when written as a $d+1$-dimensional integral but looks complicated in terms of d-forms. One can wonder whether this is also the case for $W^-_{\mathrm{c}}[v,m]$. Applying the operator $\hat{\mathsf{D}}$ to its integrand, $W^-_{\mathrm{c}}[v,m]$ can be written as a $d+1$-form; however, no simplification occurs. Again, a large simplification would have been in contradiction with the unambiguous form of the effective current.

5 Further comments and results

5.1 The chiral circle constraint

The previous calculations are completely general as regards the chiral group and the external field configurations, since no assumption has been made on the algebraic properties in flavor space. Let us now discuss the form of the functional on the chiral circle. A field configuration (v,m) is on the chiral circle when $m_{\mathrm{LR}}(x) = MU(x)$ and $m_{\mathrm{RL}}(x) = MU^{-1}(x)$ where M is a constant c-number. By unitarity U must be a unitary matrix, but in practice we will only use that U is nowhere singular. Due to dimensional counting M cannot appear in $W^-[v,m]$ (since we are considering the leading term only and all dimensions are already accounted for by the derivatives and the gauge fields); thus we can take $M = 1$ and express the chiral circle constraint as $m_{\mathrm{LR}}m_{\mathrm{RL}} = 1$ or equivalently as $\mathsf{m}^2 = 1$.

As is well known, on the chiral circle the leading term of $W^-[v,m]$ is saturated by the gauged WZW action $\mathit{\Gamma}_{\mathrm{LR}}[v,U]$. This comes about because it is possible to chirally rotate the configuration by U to bring it to the form $m_{\mathrm{LR}} = m_{\mathrm{RL}} = 1$ and so $W^-[v,m]$ is given by $\mathit{\Gamma}_{\mathrm{LR}}[v,U]$ plus $W^-_{\mathrm{VA}}[v,m=1]$ (see Appendix A). Thus the statement is equivalent to saying that the leading term of $W^-_{\mathrm{VA}}[v,m]$ vanishes when $m_{\mathrm{LR}} = m_{\mathrm{RL}} = 1$. This follows because the possible vector gauge invariant terms constructed out of v_{L} and v_{R} of dimension d vanish identically. (Vector gauge invariance is the remaining chiral invariance compatible with the condition $m_{\mathrm{LR}} = m_{\mathrm{RL}} = 1$, and it must be preserved, since all the anomaly is saturated by the gauged WZW term.)

The fact that, on the chiral circle at leading order in the derivative expansion, $W^-[v,m] = \mathit{\Gamma}_{\mathrm{LR}}[v,U]$ is of course contained in our general formulas. Actually, a stronger statement can be deduced, namely, the leading order of $W^-[v,m]$ is saturated by $\mathit{\Gamma}_{\mathrm{LR}}[v,U]$ whenever

$$m_{\mathrm{LR}}(x) = M(x)U(x), \quad m_{\mathrm{RL}}(x) = M(x)U^{-1}(x), \tag{104}$$

where $M(x)$ is a c-number but not necessarily a constant. (Note that this class of configurations is closed under chiral transformations.) To show this, let us define the symbol M by $(\mathsf{M})_{\mathrm{RR}} = (\mathsf{M})_{\mathrm{LL}} = M$ and U by $(\mathsf{U})_{\mathrm{LR}} = U$ and $(\mathsf{U})_{\mathrm{RL}} = U^{-1}$. (Note that $\mathsf{U}^2 = 1$ and so $\mathsf{U} = \mathsf{U}^{-1}$.) This allows us to use Convention 2: $\mathsf{m} = \mathsf{MU}$ and $\mathsf{m}' = \mathrm{d}\mathsf{MU} + \mathsf{MU}'$.

Consider first the extended WZW term. Clearly, in $\mathit{\Gamma}_{\mathrm{gWZW}}[v,m]$ all dependence on M without derivatives cancels, by simple dimensional counting. Likewise, all terms with two or more $\mathrm{d}\mathsf{M}$ also cancel trivially since $\mathrm{d}\mathsf{M}$ is a c-number and $(\mathrm{d}\mathsf{M})^2 = 0$. Finally, the terms linear in $\mathrm{d}\mathsf{M}$ can be shown to cancel too, by using (recall that $\mathsf{U} = \mathsf{U}^{-1}$)

$$\mathsf{R}_{\mathrm{c}} = \mathsf{R}^U_{\mathrm{c}} + \mathsf{R}_M, \quad \mathsf{L}_{\mathrm{c}} = \mathsf{R}^U_{\mathrm{c}} - \mathsf{R}_M, \quad \mathsf{R}^U_{\mathrm{c}} = \mathsf{UU}', \quad \mathsf{R}_M = \mathsf{M}^{-1}\mathrm{d}\mathsf{M}. \tag{105}$$

For instance $\mathsf{R}^5_{\mathrm{c}} = (\mathsf{R}^U_{\mathrm{c}})^5 + (\mathsf{R}^U_{\mathrm{c}})^4\mathsf{R}_M$ and $\mathsf{L}^5_{\mathrm{c}} = (\mathsf{R}^U_{\mathrm{c}})^5 - (\mathsf{R}^U_{\mathrm{c}})^4\mathsf{R}_M$; thus $\mathsf{R}^5_{\mathrm{c}} + \mathsf{L}^5_{\mathrm{c}} = 2(\mathsf{R}^U_{\mathrm{c}})^5$. Therefore, when $\mathsf{m} = \mathsf{MU}$,

$$\mathit{\Gamma}_{\mathrm{gWZW}}[v,m] = \mathit{\Gamma}_{\mathrm{LR}}[v,U]; \tag{106}$$

this is the same as the chiral circle result.

Let us now show that $W^-_{\mathrm{c}}[v,m]$ vanishes identically. In fact, this holds not only for the true functional $W^-_{\mathrm{c}}[v,m]$

but also for any other finite functional with the correct symmetries. Therefore, only general properties of the functions N are needed in the proof.

Consider first the term with N_{12}. Due to scale invariance

$$N(\mathsf{m}_1,\mathsf{m}_2)=\frac{1}{\mathsf{M}^2}N(\mathsf{U}_1,\mathsf{U}_2). \tag{107}$$

Because $\mathsf{U}^2=1$, the function $N(\mathsf{U}_1,\mathsf{U}_2)$ is completely equivalent to one where each of the $\mathsf{U}_{1,2}$ is raised to the first power at most:

$$N(\mathsf{U}_1,\mathsf{U}_2)=a+b\mathsf{U}_1+c\mathsf{U}_2+d\mathsf{U}_1\mathsf{U}_2, \tag{108}$$

where a,b,c,d are some constants (these constants exist since N_{12} is finite in the coincidence limit). However, consistency requires that $b=c=0$ (N_{12} is an even function of m). Further, mirror symmetry requires $a=d=0$ too, and $W_{\mathrm{c}}^-[v,m]$ vanishes identically in two dimensions for configurations of the form $\mathsf{m}=\mathsf{MU}$.

In four dimensions, scale invariance, consistency and the cyclic property imply

$$\begin{aligned}&N(\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3,\mathsf{m}_4)\\&\quad=\frac{1}{\mathsf{M}^4}(a+b(\mathsf{U}_1\mathsf{U}_2+\mathsf{U}_2\mathsf{U}_3+\mathsf{U}_3\mathsf{U}_4-\mathsf{U}_4\mathsf{U}_1))\end{aligned} \tag{109}$$

for some constants a and b. However, mirror symmetry requires $a=b=0$. Thus, there is no contribution from $\langle N_{1234}\mathsf{m}'^4\rangle$.

The term $\langle N_{123}\mathsf{m}'^2\mathsf{F}\rangle$ is slightly more complicated. In this case, scale invariance, consistency and mirror symmetry imply

$$N(\mathsf{m}_1,\mathsf{m}_2,\mathsf{m}_3)=\frac{a}{\mathsf{M}^2}(\mathsf{U}_1-\mathsf{U}_3)\mathsf{U}_2. \tag{110}$$

The constant a needs not vanish (in fact, $a=-1/2$ for the true functional). Nevertheless, a straightforward calculation using $\mathsf{m}'=\mathsf{dMU}+\mathsf{MU}'$, $\mathsf{UU}'=-\mathsf{U}'\mathsf{U}$ and that M is a c-number, shows that this contribution vanishes as well.

It is also worth point out that on the strict chiral circle, i.e. M constant, $W_{\mathrm{c}}^-[v,m]$ can be shown to vanish without assuming mirror symmetry.

Another remark is that the previous statements also hold for any Abelian theory, i.e. when all matrices are c-numbers in flavor space. This is because in this case m_{LR} and m_{RL} can certainly be written as in (104) with $M(x)$ a c-number, so the previous results apply.

After all these null results, one could wonder whether the chiral invariant remainder is not actually identically zero, although this is not obvious due to the notation. We have explicitly verified that this is not the case using a two-flavor model in two dimensions without accidental symmetries.

5.2 The effective density

In this subsection we give explicit formulas for the effective density J_m^- introduced in (34) as the variation of the effective action with respect to m. The general form of the densities is

$$\begin{aligned}\mathsf{J}^-_{m,d=2}&=B_{123}\mathsf{m}'^2+B_{12}\mathsf{F},\\ \mathsf{J}^-_{m,d=4}&=B_{12345}\mathsf{m}'^4+B_{1234}\mathsf{m}'\mathsf{F}\mathsf{m}'+B'_{1234}\mathsf{m}'^2\mathsf{F}\\&\quad+B''_{1234}\mathsf{F}\mathsf{m}'^2+B'_{123}\mathsf{F}^2.\end{aligned} \tag{111}$$

For consistency, the functions B are all odd under $\mathsf{m}\to-\mathsf{m}$. In addition, mirror symmetry implies

$$\begin{aligned}&B_{12}=B_{21},\quad B_{123}=B_{321},\quad B'_{123}=B'_{321},\\&B_{1234}=B_{4321},\quad B'_{1234}=B''_{4321},\quad B_{12345}=B_{54321}\end{aligned} \tag{112}$$

The effective density can be computed from scratch, by the same method used in Sect. 3.2 for the effective current. Within our approach, the direct calculation of the density is harder than for the current, because they are of higher order (J_m^- is a d-form whereas J_v^- is a $d-1$-form). A better procedure is to obtain the effective density as the variation of the effective action, which has already been computed. As we noted, the consistent effective density is also covariant.

An explicit variation of $\mathit{\Gamma}_{\mathrm{gWZW}}[v,m]$ and $W_{\mathrm{c}}^-[v,m]$ in two dimensions yields

$$\begin{aligned}B_{12}&=-2(\mathsf{m}_1-\mathsf{m}_2)N_{12}+\frac{1}{\mathsf{m}_1}+\frac{1}{\mathsf{m}_2},\\ B_{123}&=2\left(\frac{N_{13}-N_{23}}{\mathsf{m}_1-\mathsf{m}_2}-\frac{N_{12}-N_{13}}{\mathsf{m}_2-\mathsf{m}_3}+\frac{N_{23}-N_{12}}{\mathsf{m}_3+\mathsf{m}_1}\right)\\&\quad-\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3}.\end{aligned} \tag{113}$$

The terms with N are those coming from $W_{\mathrm{c}}^-[v,m]$; the other come from $\mathit{\Gamma}_{\mathrm{gWZW}}[v,m]$.

In four dimensions,

$$\begin{aligned}B'_{123}&=-(\mathsf{m}_1-\mathsf{m}_2)N_{\underline{3}12}-(\mathsf{m}_2-\mathsf{m}_3)N_{23\underline{1}}-\frac{2}{\mathsf{m}_3}\\&\quad-\frac{1}{\mathsf{m}_2}-\frac{2}{\mathsf{m}_1}-\frac{\mathsf{m}_2}{\mathsf{m}_1\mathsf{m}_3},\\ B_{1234}&=4(\mathsf{m}_2-\mathsf{m}_3)N_{1234}-\frac{N_{\underline{3}12}-N_{\underline{4}12}}{\mathsf{m}_3-\mathsf{m}_4}\\&\quad+\frac{N_{34\underline{1}}-N_{34\underline{2}}}{\mathsf{m}_1-\mathsf{m}_2}+\frac{N_{34\underline{2}}-N_{\underline{3}12}}{\mathsf{m}_4+\mathsf{m}_1}\\&\quad+\frac{1}{\mathsf{m}_1\mathsf{m}_3\mathsf{m}_4}+\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_4},\\ B'_{1234}&=-4(\mathsf{m}_3-\mathsf{m}_4)N_{1234}+\frac{N_{\underline{4}13}-N_{\underline{4}23}}{\mathsf{m}_1-\mathsf{m}_2}\\&\quad-\frac{N_{\underline{4}12}-N_{\underline{4}13}}{\mathsf{m}_2-\mathsf{m}_3}+\frac{N_{\underline{4}23}-N_{123}}{\mathsf{m}_4+\mathsf{m}_1}\\&\quad+\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_4}+\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3},\\ B_{12345}&=4\left(\frac{N_{1345}-N_{2345}}{\mathsf{m}_1-\mathsf{m}_2}-\frac{N_{1245}-N_{1345}}{\mathsf{m}_2-\mathsf{m}_3}\right.\\&\quad+\frac{N_{1235}-N_{1245}}{\mathsf{m}_3-\mathsf{m}_4}-\frac{N_{1234}-N_{1235}}{\mathsf{m}_4-\mathsf{m}_5}\\&\quad\left.+\frac{N_{2345}-N_{1234}}{\mathsf{m}_5+\mathsf{m}_1}\right)-\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4\mathsf{m}_5}.\end{aligned} \tag{114}$$

There is an alternative way to obtain the density, which is simpler and also serves as a check, namely by using the anomaly equation. This is (34) when the variations are associated to an infinitesimal chiral rotation, see (8),

$$\hat{\mathsf{D}}\mathsf{J}_v^- + \{\mathsf{J}_m^-, \mathsf{m}\} = \mathcal{A}, \tag{115}$$

where $\mathcal{A}$ is the consistent chiral anomaly (defined so that $\delta W_{\mathrm{LR}}^-[v,m] = \langle \mathcal{A}\alpha \rangle$ is the left-hand side of (A1)). Note that J_v^- is the consistent current. The contribution of the counterterm current $\mathsf{P}(v)$ cancels the chiral symmetry breaking terms from the anomaly. This yields the following formulas for the density

$$\begin{aligned}
B_{12} &= \frac{1}{\mathsf{m}_1+\mathsf{m}_2}(4+(\mathsf{m}_1-\mathsf{m}_2)A_{12}),\\
B_{123} &= -\frac{1}{\mathsf{m}_1+\mathsf{m}_3}(\Delta A)_{123},\\
B'_{123} &= -\frac{1}{\mathsf{m}_1+\mathsf{m}_3}(12-(\mathsf{m}_2-\mathsf{m}_3)A_{123}\\
&\quad +(\mathsf{m}_1-\mathsf{m}_2)A_{321}),\\
B_{1234} &= -\frac{1}{\mathsf{m}_1+\mathsf{m}_4}\left(\frac{A_{134}-A_{234}}{\mathsf{m}_1-\mathsf{m}_2}+\frac{A_{321}-A_{421}}{\mathsf{m}_3-\mathsf{m}_4}\right.\\
&\quad \left. +(\mathsf{m}_2-\mathsf{m}_3)A_{1234}\right),\\
B'_{1234} &= -\frac{1}{\mathsf{m}_1+\mathsf{m}_4}\left(-\frac{A_{431}-A_{432}}{\mathsf{m}_1-\mathsf{m}_2}+\frac{A_{421}-A_{431}}{\mathsf{m}_2-\mathsf{m}_3}\right.\\
&\quad \left. -(\mathsf{m}_3-\mathsf{m}_4)A_{1234}\right),\\
B_{12345} &= -\frac{1}{\mathsf{m}_1+\mathsf{m}_5}(\Delta A)_{12345}.
\end{aligned} \tag{116}$$

It can be verified that these expressions coincide with those in (113) and (114), and the possible ambiguities introduced by the functions N are explicitly removed.

As in the case of the effective current we can define a set of associated functions as follows

$$\begin{aligned}
&B_{12} = \bar{B}_{12}, \quad B_{123} = \bar{B}_{1\underline{2}3}, \quad B'_{123} = \bar{B}'_{123}, \\
&B_{1234} = \bar{B}_{1\underline{2}\underline{3}4}, \quad B'_{1234} = \bar{B}'_{1\underline{2}34}, \quad B_{12345} = \bar{B}_{1\underline{2}3\underline{4}5}.
\end{aligned} \tag{117}$$

(The rule is to flip the signs of the arguments at the right of each operator that is an odd-order differential form, in practice m'.) Once again the functions $\bar{B}$ so defined turn out to have the property of being completely symmetric under permutation of their arguments, a property already noted for the functions $\bar{A}$ of the effective current. In two dimensions this property follows solely from the general symmetries of the function N_{12}; however, in four dimensions this is not the case. Consistency and mirror symmetry follows automatically in all cases from the corresponding properties of N_{123} and N_{1234}. Invariance of $\bar{B}'_{123}$, $\bar{B}_{1234}$, and $\bar{B}'_{1234}$ under general permutations (other than mirror permutations) does not follow from the general symmetries of N_{123} and N_{1234} as is already obvious by setting these two functions to zero in (114). For $\bar{B}_{12345}$, it can be shown that invariance under cyclic permutations follows from general symmetries of N_{123} and N_{1234} but invariance under more general permutations does not. (Also the complete symmetry of the functions $\bar{A}$ combined with the formulas in (116) does not guarantee symmetry of the functions $\bar{B}$.) Therefore, the complete symmetry of the functions $\bar{B}$ in four dimensions is a specific property of the true effective action functional. Since this property is so general (it holds for effective currents and effective densities and in all dimensions examined) it is likely that it follows from the very definition of these currents rather than being an accidental symmetry.

5.3 Vector-like reduction

Our conventions for the vector–axial (VA) notation are as follows:

$$\mathrm{D} = \not{D}_V + \not{A}\,\gamma_5 + S + \gamma_5 P, \tag{118}$$

where $D_\mu^V = \partial_\mu + V_\mu$ is the vector covariant derivative and

$$v_{\mathrm{R,L}} = V \pm A, \quad m_{\mathrm{LR}} = S + P, \quad m_{\mathrm{RL}} = S - P. \tag{119}$$

Strictly speaking a purely vector-like case would mean $v_{\mathrm{R}} = v_{\mathrm{L}}$ and $m_{\mathrm{LR}} = m_{\mathrm{RL}}$, or $P = A = 0$. For such configurations there is no pseudo-parity odd component of the effective action. Thus, presently, we will refer to the case of vanishing pseudo-scalar field, $P = 0$, but not necessarily vanishing axial field A by the terminology: vector-like case. Of course, in this case it is preferable to work with the VA version of the effective action, which is related to the LR version by subtracting an appropriated m-independent counterterm (see Appendix A)

$$W_{\mathrm{VA}}^-[v,m] = W_{\mathrm{LR}}^-[v,m] - P_{\mathrm{ct}}[v]. \tag{120}$$

The counterterm is such that $W_{\mathrm{VA}}^-[v,m]$ is vector gauge invariant and the anomaly affects only axial transformations. In this subsection we will denote $W^-[v,m]$ by $W_{\mathrm{LR}}^-[v,m]$ to emphasize that it is the LR version of the effective action.

When $P = 0$, the most general form of the VA effective action (at leading order and in the pseudo-parity odd sector) is

$$\begin{aligned}
W_{\mathrm{VA},d=2}^-[v,m] &= \langle M_{12} S' A \rangle, \\
W_{\mathrm{VA},d=4}^-[v,m] &= \langle M_{123} F_V S' A + M'_{123} S' F_V A \\
&\quad + M_{1234} S'^3 A + M'_{1234} S' A^3 + M''_{123} A^2 F_A \rangle,
\end{aligned} \tag{121}$$

where $S' = [D_V, S]$, $F_V = D_V^2$ and $F_A = \{D_V, A\}$ and the various M's are functions of S, i.e, $M_{12} = M(S_1, S_2)$, etc. Note that the symbol S, unlike m, is even under cyclic permutations; thus, in particular there are no consistency restrictions on the various functions M. Also there are no cyclicity restrictions. On the other hand, mirror symmetry is guaranteed provided that

$$\begin{aligned}
&M_{12} = -M_{21}, \quad M_{123} = -M'_{321}, \quad M_{1234} = -M_{4321}, \\
&M'_{1234} = -M'_{2143}, \quad M''_{123} = -M''_{321}.
\end{aligned} \tag{122}$$

Not all these functions are unambiguously determined by the functional itself due to the following identity (the prime denotes the derivative with respect to D_V)

$$0 = \frac{1}{3}\left\langle (G_{123}A^3)' \right\rangle = \left\langle G_{123}A^2F_A + \frac{G_{134}-G_{234}}{S_1-S_2}S'A^3 \right\rangle . \tag{123}$$

Here G_{123} is any function subjected to the cyclic property restriction $G_{123} = G_{231}$. If in addition mirror symmetry is imposed, G_{123} must be a completely antisymmetric function under permutation of its arguments. This identity introduces an ambiguity because of integration by parts in M'_{1234} and M''_{123}.

Since $W^-_{\rm LR}[v,m]$ has been computed previously, (120) can be used to obtain $W^-_{\rm VA}[v,m]$. The contribution from $\Gamma_{\rm gWZW}[v,m]$ when $P=0$ is easily obtained from (71). This contribution combined with that coming from the counterterm yields

$$\begin{aligned} &W^-_{{\rm VA,WZW},d=2}[v,m] = \left\langle -[S^{-1},S']A \right\rangle , \\ &W^-_{{\rm VA,WZW},d=4}[v,m] = \Big\langle 2F_V[S^{-1},S']A + S^{-1}F_VS'A \\ &\quad -SF_VS^{-1}S'S^{-1}A + 2[S^{-1},S']F_VA - S'F_VS^{-1}A \\ &\quad +S^{-1}S'S^{-1}F_VSA - (S^{-1}S')^3A + (S'S^{-1})^3A \\ &\quad +[S^{-1},S']A^3 + S'S^{-1}ASAS^{-1}A - S^{-1}S'AS^{-1}ASA \\ &\quad -SAS^{-1}AF_A - S^{-1}ASAF_A + ASAS^{-1}F_A \\ &\quad +AS^{-1}ASF_A \Big\rangle . \end{aligned} \tag{124}$$

By construction all terms breaking vector gauge invariance have canceled.

The contribution coming from $W^-_{\rm c}[v,m]$ is also easily computed. To illustrate the method we will work out explicitly the two-dimensional case. From consistency, the most general form of N_{12} is

$$N_{12} = n'_{12} + \mathsf{m}_1\mathsf{m}_2 n_{12}, \tag{125}$$

where n_{12} and n'_{12} are functions of m_1^2 and m_2^2. (Mirror symmetry further requires n'_{12} to vanish but this will not be enforced here.) Thus (expanding Convention 2 but keeping Convention 1) yields

$$\left\langle N_{12}\mathsf{m}'^2 \right\rangle = \left\langle (n'_{12} + \mathsf{m}_1\mathsf{m}_2 n_{12})\mathsf{m}'^2 \right\rangle \tag{126}$$

$$= \left\langle n'_{12}m'_{\rm RL}m'_{\rm LR} + n_{12}(\mathsf{m}\mathsf{m}')^2_{\rm R} \right\rangle . \tag{127}$$

Now we can take $P=0$, i.e., $m_{\rm LR} = m_{\rm RL} = S$ and use the formulas

$$m'_{\rm RL} = S' + \{S,A\}, \quad m'_{\rm LR} = S' - \{S,A\}. \tag{128}$$

(Note that the prime refers to $D_{\rm R,L}$ in the left-hand side and to D_V in the right-hand side.) In addition, all arguments m become S. This gives

$$\begin{aligned} &W^-_{{\rm VA,c},d=2}[v,m] = \langle n'_{12}(S' + \{S,A\})(S' - \{S,A\}) \\ &\quad + n_{12}S_1S_2(S' - \{S,A\})(S' - \{S,A\})\rangle \\ &= \langle -n'_{12}S'\{S,A\} + n'_{12}\{S,A\}S' - n_{12}S_1S_2S'\{S,A\} \\ &\quad -n_{12}S_1S_2\{S,A\}S'\rangle \\ &= \left\langle -N_{12}S'\{S,A\} + N_{1\underline{2}}\{S,A\}S' \right\rangle \\ &= \left\langle -2N_{12}S'\{S,A\} \right\rangle . \end{aligned} \tag{129}$$

The second equality follows from Convention 1, which keeps only pseudo-parity odd terms. The last equality follows from the cyclic property of the trace. A useful observation is that, although the detailed expansion in (125) is required in intermediate steps, the full function N_{12} can be reconstructed, as in the third line above, by allowing appropriate changes in the signs of its arguments (e.g. $N_{1\underline{2}}$)[8].

In four dimensions, using

$$F_{\rm R} = F_V + A^2 + F_A, \quad F_{\rm L} = F_V + A^2 - F_A, \tag{130}$$

the result is

$$\begin{aligned} &W^-_{{\rm VA,c},d=4}[v,m] \\ &= \Big\langle N_{\underline{1}23}\{S,A\}S'(F_V + A^2) - N_{12\underline{3}}S'\{S,A\}(F_V + A^2) \\ &\quad +N_{123}S'^2F_A - N_{1\underline{2}3}\{S,A\}^2F_A - 4N_{1234}S'^3\{S,A\} \\ &\quad +4N_{12\underline{3}4}S'\{S,A\}^3 \Big\rangle . \end{aligned} \tag{131}$$

In this formula, the term S'^2F_A has to be integrated by parts in order to conform to the standard form chosen in (121).

The combination of the previous results from $\Gamma_{\rm gWZW}$ $[v,m]$, $W^-_{\rm c}[v,m]$ and $P_{\rm ct}[v]$ gives

$$\begin{aligned} M_{12} &= A_{12}, \\ M_{123} &= A_{123}, \\ M_{1234} &= A_{1234}, \\ M'_{1234} &= \frac{1}{S_1} - \frac{1}{S_2} + \frac{S_3}{S_2S_4} - \frac{S_4}{S_1S_3} - (S_2+S_3)N_{12\underline{3}} \\ &\quad - (S_1+S_4)N_{\underline{4}12} \\ &\quad + 4(S_1+S_4)(S_2+S_3)(S_3+S_4)N_{12\underline{3}4}, \\ M''_{123} &= -\frac{S_1}{S_2} - \frac{S_2}{S_1} + \frac{S_2}{S_3} + \frac{S_3}{S_2} \\ &\quad - (S_1+S_2)(S_2+S_3)N_{1\underline{2}3}. \end{aligned} \tag{132}$$

In these formulas the functions A_{12}, etc., are those of the effective current in (36).

The ambiguity in the functions N_{123} and N_{1234} translates into an ambiguity in M'_{1234} and M''_{123} of the form $G_{123} = -\widehat{H}_{1\underline{2}3}$. On the other hand, exploiting the ambiguity in these functions allows us to write explicit expressions in terms of the A's, as follows:

$$\begin{aligned} M'_{1234} &= -\frac{1}{3}\frac{(S_1+S_4)A_{143} - (S_2+S_4)A_{243}}{S_1-S_2} \\ &\quad - \frac{1}{3}\frac{(S_1+S_3)A_{\underline{1}43} - (S_2+S_3)A_{\underline{2}43}}{S_1-S_2} \\ &\quad - (S_1+S_4)(S_2+S_3)A_{12\underline{34}}, \\ M''_{123} &= \frac{1}{3}(S_1+S_2)A_{3\underline{1}2} - \frac{1}{3}(S_2+S_3)A_{1\underline{3}2}. \end{aligned} \tag{133}$$

[8] The empirical rule is to flip the signs of the arguments at the right of each A or F_A. In addition there is a global minus sign for each A or F_A occupying an even position.

Note that these functions differ from those in (132), although, of course, they produce the same functional.

An alternative way to obtain the functions M is based on reproducing the correct axial current. This method yields (133) more directly. The procedure is straightforward, so we do not give details; however, it is worth noticing that with our notation J_v^- denotes simultaneously the left and right currents and the vector and axial currents (all of them associated to the LR version of the effective action). The chiral currents are defined by

$$J^-_{v,\mathrm{R}} = (\mathsf{J}^-_v)_\mathrm{R}, \quad J^-_{v,\mathrm{L}} = (\mathsf{J}^-_v)_\mathrm{L}, \tag{134}$$

so that (consistently with Convention 1)

$$\delta W^-_{\mathrm{LR}}[v,m] = \frac{1}{2}\langle J^-_{v,\mathrm{R}}\delta v_\mathrm{R} - J^-_{v,\mathrm{L}}\delta v_\mathrm{L}\rangle. \tag{135}$$

On the other hand, the vector and axial currents are defined by

$$\delta W^-_{\mathrm{LR}}[v,m] = \langle J^-_V \delta V + J^-_A \delta A\rangle. \tag{136}$$

Thus

$$J^-_{V,A} = \frac{1}{2}(J^-_{v,\mathrm{R}} \mp J^-_{v,\mathrm{L}}) = \mathsf{J}^-_v\,. \tag{137}$$

In the last equality we are using our conventions with the proviso that J_V^- and J_A^- are pseudo-parity odd and even quantities, respectively. (Of course the usual vector and axial current are those associated to the VA version of the effective action, so it still remains for us to pick up the contribution from the counterterm $P_{\mathrm{ct}}[v]$.)

5.4 The two-dimensional pseudo-parity odd effective action from the anomaly

In this subsection we will point out a general property of the effective action in the pseudo-parity odd sector, which holds to all orders and for any gauge group and any spacetime dimension greater than zero, and we will show that this property is sufficient to completely fix $W^-[v,m]$ at leading order in two dimensions from the chiral anomaly.

The general property is that $W^-[v,m]$ vanishes identically when there are no gauge fields and one of the scalar fields, say m_{RL}, is a spacetime constant, that is,

$$W^-[v,m_{\mathrm{LR}},m_{\mathrm{RL}}] = 0, \quad \text{when} \quad v=0, \quad \mathrm{d}m_{\mathrm{RL}} = 0. \tag{138}$$

To proof this statement, let us consider the variation of $W^-[v,m]$ within this class of configurations when only m_{LR} is varied. Use of (52) yields

$$\begin{aligned}\delta W^-[v,m] &= -\frac{1}{2}\mathrm{Tr}\left[\gamma_5\frac{1}{m_{\mathrm{LR}} - \not\partial\, m_{\mathrm{RL}}^{-1}\not\partial}\delta m_{\mathrm{LR}}\right] \\ &= -\frac{1}{2}\mathrm{Tr}\left[\gamma_5\frac{1}{m_{\mathrm{LR}} - m_{\mathrm{RL}}^{-1}\partial^2}\delta m_{\mathrm{LR}}\right] \\ &= 0. \end{aligned} \tag{139}$$

The second equality holds due to $\mathrm{d}m_{\mathrm{RL}} = 0$. The last equality follows from $\mathrm{tr}\gamma_5 = 0$ (except at $d=0$, and indeed the property does not hold in this case). Therefore the value of $W^-[v=0, m_{\mathrm{LR}}, m_{\mathrm{RL}} = \text{constant}]$ does not depend on m_{LR}. This value is zero as follows from choosing $m_{\mathrm{LR}} = m_{\mathrm{RL}}$, since in this case the configuration is unchanged under pseudo-parity conjugation and the pseudo-parity odd component vanishes. Note that this property is specific of the effective action functional and does not derive from general symmetry properties of this functional. From (70), it follows that within this class of configurations

$$W^-_\mathrm{c}[v,m] = -\Gamma_{\mathrm{gWZW}}[v,m] \quad (v=0, \quad \mathrm{d}m_{\mathrm{RL}}=0). \tag{140}$$

All higher orders in the derivative expansion must vanish separately, whereas the leading term of $W^-_\mathrm{c}[v,m]$ must cancel the extended gauged WZW term.

Next we will use this property to determine the chiral covariant remainder in two dimensions. To do this let us compute the two sides of (140) when $v=0$, $m_{\mathrm{RL}}=1$ (or any constant c-number) and $m_{\mathrm{LR}} = \mu$ (this is just a change of name). Using $(\mathsf{m}^{-1}\mathrm{d}\mathsf{m})_\mathrm{R} = \mu^{-1}\mathrm{d}\mu$, and $(\mathsf{m}^{-1}\mathrm{d}\mathsf{m})_\mathrm{L} = 0$, one finds

$$\begin{aligned}\Gamma_{\mathrm{gWZW}}[v,m] &= \frac{1}{2}\Gamma_{\mathrm{LR}}[v=0, U=\mu] \\ &= \frac{1}{2}\left\langle -\frac{1}{3}(\mu^{-1}\mathrm{d}\mu)^3\right\rangle \\ &= \frac{1}{2}\left\langle h_{\mathrm{WZW}}(\mu_1,\mu_2)\mathrm{d}\mu^2\right\rangle, \end{aligned} \tag{141}$$

where the function $h_{\mathrm{WZW}}(z_1,z_2)$ was introduced in (26).

On the other hand, using only symmetry arguments (including analyticity of the effective action functional) the leading term of $W^-_\mathrm{c}[v,m]$ in two dimensions must have the form given in (79) with

$$N(\mathsf{m}_1,\mathsf{m}_2) = \mathsf{m}_1\mathsf{m}_2 n(\mathsf{m}_1^2,\mathsf{m}_2^2), \tag{142}$$

for some antisymmetric function $n(z_1,z_2)$ to be determined. For the class of configurations selected above, and using the fact that in this case $m_\mathrm{L}^2 = m_\mathrm{R}^2 = \mu$, $(\mathsf{m}\mathrm{d}\mathsf{m})_\mathrm{R} = \mathrm{d}\mu$, $(\mathsf{m}\mathrm{d}\mathsf{m})_\mathrm{L} = 0$, one finds

$$W^-_\mathrm{c}[v,m] = \frac{1}{2}\left\langle n(\mu_1,\mu_2)\mathrm{d}\mu^2\right\rangle. \tag{143}$$

Comparing both calculations, it follows that $n(z_1,z_2) = -h_{\mathrm{WZW}}(z_1,z_2)$ and thus

$$N(\mathsf{m}_1,\mathsf{m}_2) = -\mathsf{m}_1\mathsf{m}_2 h_{\mathrm{WZW}}(\mathsf{m}_1^2,\mathsf{m}_2^2), \tag{144}$$

which is indeed verified by the correct function N_{12} given in (92).

The points to remark are:
(i) since the WZW term is completely determined by integration of the chiral anomaly, the function $h_{\mathrm{WZW}}(z_1,z_2)$ also follows from the anomaly;
(ii) although W^-_c is considered for a particular case, this is sufficient to determine the function N_{12} because no special properties of μ (i.e. particular flavor groups) have been assumed.

In four dimensions this method is insufficient to fix the effective action. The function N_{123} does not contribute since $\mathsf{F}=0$ when we take $v=0$. On the other hand, N_{1234} can be decomposed as

$$\begin{aligned} N_{1234} = n_{1234} + \mathsf{m}_1\mathsf{m}_2 n'_{1234} + \mathsf{m}_2\mathsf{m}_3 n'_{2341} + \mathsf{m}_3\mathsf{m}_4 n'_{3412} \\ -\mathsf{m}_1\mathsf{m}_4 n'_{4123} + \mathsf{m}_1\mathsf{m}_3 n''_{1234} + \mathsf{m}_2\mathsf{m}_4 n''_{2341} \\ +\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4 n'''_{1234}, \end{aligned} \quad (145)$$

where the various n's are functions of m^2. For configurations with $v=0$ and $m_{\mathrm{RL}}=1$, only the last component n'''_{1234} gives a contribution, hence all other components remain undetermined by this procedure. The component n'''_{1234} is fixed by imposing a cancellation with the contribution coming from $\Gamma_{\mathrm{gWZW}}[v,m]$. We have explicitly verified this with our formulas. In passing, we note another unexpected property, namely, the component n_{1234} vanishes identically, although this is not required by the general symmetries of N_{1234}. It seems to be a specific property of the effective action functional (this statement depends only of our choice of $\Gamma_{\mathrm{gWZW}}[v,m]$). The other components do not vanish identically.

To finish this subsection, let us note that the same observation and method described above can be adapted to the VA version of the effective action. The fact that $W^-_{\mathrm{LR}}[v,m]$ vanishes when $v=0$ and m_{RL} is a constant, (138), implies (making a chiral rotation to the case $P=0$ and using the formulas in Appendix A) that

$$\begin{aligned} W^-_{\mathrm{VA}}[v,m] = -\Gamma_{\mathrm{WZW}}(U), \quad \text{when} \\ v_{\mathrm{R,L}} = U^{\pm 1/2}\mathrm{d}U^{\mp 1/2}, \quad m_{\mathrm{LR}} = m_{\mathrm{RL}} = U. \end{aligned} \quad (146)$$

This identity can then be used to determine the function M_{12} in $W^-_{\mathrm{VA}}[v,m]$ in two dimensions. A straightforward calculation yields

$$M(z_1,z_2) = \frac{4z_1z_2}{z_1+z_2} h_{\mathrm{WZW}}(z_1,z_2). \quad (147)$$

This relation is verified by the correct function $M_{12}=A_{12}$. Combining this formula and that in (144) yields

$$N(\mathsf{m}_1,\mathsf{m}_2) = -\frac{\mathsf{m}_1^2+\mathsf{m}_2^2}{4\mathsf{m}_1\mathsf{m}_2} A(\mathsf{m}_1^2,\mathsf{m}_2^2), \quad (148)$$

which is a non-trivial relation between the covariant current, A_{12}, and the covariant remainder, N_{12}.

5.5 Further properties of the extended gauged WZW action

The functional $\Gamma_{\mathrm{gWZW}}[v,m]$ in (70) is required to reproduce the correct chiral anomaly, but otherwise it is a matter of choice. A different choice would be compensated by a change in the chiral invariant remainder. Nevertheless, the concrete form proposed in (71) is the unique such functional enjoying two further properties, namely,

(i) it does not mix m_{LR} with m_{RL}, and

(ii) it is invariant under the transformation $\mathsf{m}\to\mathsf{m}^{-1}$ (i.e. $m_{\mathrm{LR}} \leftrightarrow m^{-1}_{\mathrm{RL}}$). The second property is manifest in (75) for the gauged terms. For the WZW term it holds too:

$$\begin{aligned} \langle \mathsf{R}^{d+1}\rangle \to \langle \mathsf{L}^{d+1}\rangle &= \langle (-\mathsf{m}\mathsf{R}\mathsf{m}^{-1})^{d+1}\rangle \\ &= -\langle \mathsf{m}\mathsf{R}^{d+1}\mathsf{m}^{-1}\rangle \\ &= \langle \mathsf{R}^{d+1}\rangle . \end{aligned} \quad (149)$$

That $\Gamma_{\mathrm{gWZW}}[v,m]$ is fully characterized by these two properties can be seen after a detailed analysis: any other such functional would differ by a chiral invariant contribution, of the same form as $W^-_{\mathrm{c}}[v,m]$ in (79). The requirement of not mixing m_{LR} and m_{RL} only allows $a\langle \mathsf{R}_{\mathrm{c}}^4\rangle$ for the term with N_{1234} (e.g., a piece m^2 introduces a mixing, and similarly $\mathsf{m}\mathsf{m}'$ or m'^2), and such a term vanishes identically. For the term with N_{123}, the most general form not mixing m_{LR} and m_{RL} would be $\langle a\mathsf{R}_{\mathrm{c}}^2\mathsf{F} + b\mathsf{L}_{\mathrm{c}}^2\mathsf{F}\rangle$; however, mirror symmetry requires $b=-a$ and this in conflict with invariance under $\mathsf{m}\to\mathsf{m}^{-1}$, which requires $b=a$.

As noted above, the property of not mixing m_{LR} and m_{RL} does not extend to the full effective action. Let us discuss the property of invariance under $\mathsf{m}\to\mathsf{m}^{-1}$. First of all, note that it cannot be a symmetry of the effective action beyond the leading term in the derivative expansion, since it does not preserve the dimensional counting, so our next comments refer to this leading term only (for W^- or the term with precisely d derivatives for W^+).

On the chiral circle, the transformation $\mathsf{m}\to\mathsf{m}^{-1}$ is a trivial symmetry (since $\mathsf{m}^2=1$). As we have just seen, it is also a symmetry of the functional $\Gamma_{\mathrm{gWZW}}[v,m]$ (on or off the chiral circle). Remarkably, it turns out to be an invariance of the leading term of $W^-[v,m]$ in zero and two dimensions. In the zero-dimensional case this is obvious since $W^-_{\mathrm{c}}[v,m]$ vanishes. In two dimensions it is an accidental symmetry which follows as an automatic consequence of chiral and Lorentz invariance, plus scale invariance and mirror symmetry. Indeed, the most general form of N_{12} consistent with scale invariance and mirror symmetry is

$$N(\mathsf{m}_1,\mathsf{m}_2) = \frac{1}{\mathsf{m}_1\mathsf{m}_2} f(\mathsf{m}_1\mathsf{m}_2^{-1}), \quad f(x) = -f(x^{-1}). \quad (150)$$

On the other hand, under the transformation $\mathsf{m}\to\mathsf{m}^{-1}$, $\mathsf{m}'\to -\mathsf{m}^{-1}\mathsf{m}'\mathsf{m}^{-1}$ and so

$$N(\mathsf{m}_1,\mathsf{m}_2) \to -\frac{1}{\mathsf{m}_1^2\mathsf{m}_2^2} N(\mathsf{m}_1^{-1},\mathsf{m}_2^{-1}) = N(\mathsf{m}_1,\mathsf{m}_2). \quad (151)$$

The same invariance is also automatic in the term with two covariant derivatives in W^+ in two dimensions (although it fails in zero dimensions for W^+). In four dimensions such a transformation is not a symmetry of the leading term of $W^-[v,m]$, as can be seen using the explicit formula of N_{123} in Appendix B or A_{123} for the effective current.

5.6 General form of the chiral invariant remainder

In Sect. 4.2 we have noted that the forms taken in (79) for $W^-_{\mathrm{c}}[v,m]$ in two and four dimensions are actually the most general ones that form those functionals. To show this, let us begin by considering a functional of the form $\langle N_1 \mathsf{F}\rangle$ in two dimensions. Using the identity

$$\begin{aligned} 0 = \langle (f_1 \mathsf{m}')' \rangle &= \langle (\Delta f)_{12} \mathsf{m}'^2 + f_1 \mathsf{m}'' \rangle \\ &= \langle (\Delta f)_{12} \mathsf{m}'^2 + f_1 [\mathsf{F}, \mathsf{m}] \rangle \\ &= \langle (\Delta f)_{12} \mathsf{m}'^2 - 2\mathsf{m} f_1 \mathsf{F} \rangle, \end{aligned} \quad (152)$$

it follows that $\langle N_1 \mathsf{F}\rangle$ can be reabsorbed in $\langle N_{12}\mathsf{m}'^2\rangle$ by taking $f_1 = -(1/2)N_1$.

In four dimensions, using $\mathsf{m}'' = [\mathsf{F}, \mathsf{m}]$ and $\mathsf{F}' = 0$, the most general form is that given in (79) augmented with terms of the form $\langle N'_{12}\mathsf{F}^2\rangle$. However, due to mirror symmetry

$$N'_{12} = -N'_{21}, \quad N'_{12} = (\mathsf{m}_1 - \mathsf{m}_2) n''_{12}; \quad (153)$$

that is, the function $n''_{12} = N'_{12}/(\mathsf{m}_1 - \mathsf{m}_2)$ is finite (in the coincidence limit) if N'_{12} is finite. Then,

$$\begin{aligned} \langle N'_{12}\mathsf{F}^2\rangle &= \langle n''_{12}[\mathsf{m}, \mathsf{F}]\mathsf{F}\rangle = \langle -n''_{12}\mathsf{m}''\mathsf{F}\rangle \\ &= \langle (\Delta n'')_{123}\mathsf{m}'^2\mathsf{F} + (-n''_{12}\mathsf{m}'\mathsf{F})'\rangle \\ &= \langle (\Delta n'')_{123}\mathsf{m}'^2\mathsf{F}\rangle. \end{aligned} \quad (154)$$

Therefore, the term $\langle N'_{12}\mathsf{F}^2\rangle$ is also redundant.

Let us now discuss the existence of more general chiral invariant functionals which do not have the analytical form in (79). Since the functional is chiral invariant, it can be computed in a chirally rotated configuration. The point is that it is always possible to chirally rotate a configuration so that $m_{\rm RL} = m_{\rm LR} = S$, $P = 0$. In the chiral gauge $P = 0$ the only remaining freedom is that of vector gauge transformations. Therefore, there are as many chiral invariant functionals of $m_{\rm LR}$, $m_{\rm RL}$, $v_{\rm R}$ and $v_{\rm L}$ as there are vector gauge invariant functionals of S, V and A. These rotated VA fields depend on the original chiral fields in a non-analytical way.

The most general VA functional has been considered in Sect. 5.3, (121). In four dimensions (and assuming mirror symmetry) it depends on four independent functions: M_{123}, M_{1234}, M'_{1234} and M''_{123}. When the functional derives from an analytical form, these functions take the form given in (132). In particular, M_{123}, and M_{1234}, coincide with the functions A_{123} and A_{1234} of the effective currents. (The gWZW contribution has to be removed from these functions, but this does not change the argument.) Since the current determines the effective action, it follows that M_{123} and M_{1234} determine N_{123} and N_{1234}, and so determine the other two functions M'_{1234} and M''_{123}. This already implies that the analytical form is not the most general one, since one could imagine new functionals obtained by keeping the same M_{123} and M_{1234}, but arbitrarily modifying M'_{1234} and M''_{123}. Such functionals would not be equivalent to an analytical one for any choice of N_{123} and N_{1234}.

Even in two dimensions, where the VA functional contains only one arbitrary function M_{12}, the analytical functional $\langle N_{12}\mathsf{m}'^2\rangle$ is not the most general one. When the VA functional is analytical (in terms of the unrotated variables)

$$M_{12} = -2(S_1 + S_2)N_{12}. \quad (155)$$

(This is just (90) removing the gWZW contribution. N_{12} is evaluated at $\mathsf{m}_{1,2} = S_{1,2}$.) The function N_{12} is restricted by consistency, cyclic symmetry and finiteness (we do not enforce mirror symmetry here) and this implies

$$M_{12} = -M_{\underline{1}2}, \quad (S_1 - S_2)M_{12} = (S_1 + S_2)M_{\underline{2}1}, \quad M_{1\underline{1}} = 0. \quad (156)$$

If M_{12} is analytical in S, the first condition follows from dimensional counting (unless the VA functional breaks scale invariance or depends on new external fields), but the other two conditions are not required to have an acceptable VA functional (M_{12} still has to be finite at $S_1 = S_2$). For instance,

$$\Gamma[S, V, A] = \left\langle \frac{1}{S} S' A \right\rangle \quad (157)$$

violates the conditions and so it cannot be written as $\langle N_{12}\mathsf{m}'^2\rangle$ for any suitable N_{12}.

Another comment is the following. At the end of Sect. 2.2 we noted that one could consider phenomenological contributions of the form $\langle h(u_1, u_2) {\rm d}u^2\rangle$ in two dimensions (and similar comments apply to four dimensions as well) which are consistent with vector gauge invariance but are not chiral invariant except when the true function $h_{\rm WZW}(u_1, u_2)$ is used. Chiral invariance in no longer a problem for functionals of the form $\langle h(\mathsf{m}_1, \mathsf{m}_2)\mathsf{m}'^2\rangle$ (i.e. the same form of $W^-_{\rm c}[v, m]$ but with a different function). Such phenomenological terms, which are vanishing on the chiral circle, are topological in the sense that they do not contribute the strength-energy tensor and their corresponding baryonic current is conserved independently of the equations of motion. This can be seen from (115): setting $v = 0$ and taking the trace it says that the baryonic current is a closed form, and this result does not depend of the explicit form of $W^-_{\rm c}[v, m]$.

5.7 Descent relations

It is known that the VA version of the pseudo-parity odd component of the effective action equals $2\pi{\rm i}$ times the baryon number in two more dimensions [19,20,11] (see [14] for a proof in the framework of the ζ-function regularized effective action). In this relation one of the extra dimensions, u, is regarded as the time and the other, v, is a new space direction. The relation holds provided that the dependence of the $d + 2$-dimensional configuration is u-independent and adiabatic in v, so that no more than one v derivative is retained. In this case, and choosing $\eta_d = {\rm i}^{d/2}$, the relation takes the form

$$W^-_{{\rm VA},d} = -2\pi{\rm i}\langle \mathsf{J}^-_{{\rm VA},V}\rangle_{d+2} \quad (\eta_d = {\rm i}^{d/2}). \quad (158)$$

The subscripted dimension in the right-hand side refers to the normalization of $\langle\ \rangle$, (3). $\mathsf{J}^-_{{\rm VA},V}$ denotes the vector current associated to $W^-_{{\rm VA},d+2}[v, m]$. Under the conditions stated above, and due to gauge invariance, only the pseudo-parity odd component of the current has a contribution to the baryon number [14].

The previous relation can be rewritten as one for the LR version as follows:

$$W_d^-[v,m] = -2\pi\mathrm{i}\langle \mathsf{J}^-_{v,\mathrm{c}}\rangle_{d+2} - 2W_{\mathrm{CS},d+1}[v], \qquad (159)$$

where W_{CS} is the Chern–Simons action

$$W_{\mathrm{CS},d=1}[v] = \langle \mathsf{v}\rangle_{d=0}\,,$$
$$W_{\mathrm{CS},d=3}[v] = \left\langle \frac{1}{3}\mathsf{v}^3 - \mathsf{vF}\right\rangle_{d=2}. \qquad (160)$$

The Chern–Simons terms are precisely those appearing in (75), and account for all the chiral symmetry breaking in $W_d^-[v,m]$. (The factor of 2 in W_{CS} accounts for a Chern–Simons term for the right field and the other for the left field.)

Let us give details on the derivation of (159) for $d=2$ (assuming (158)). The four-dimensional counterterm relating the VA and LR versions is given in (A6). Its contribution to the vector current is minus

$$J_{\mathrm{ct},d=4} = -2\{\mathsf{F},\mathsf{v}\} + 2\mathsf{v}^3 + 6\mathrm{d}(v_{\mathrm{R}}v_{\mathrm{L}}). \qquad (161)$$

This contribution is to be combined in (158) with that of the counterterm current, relating the consistent and covariant currents, (87). This yields

$$2\pi\mathrm{i}\,\langle \mathsf{P} - J_{\mathrm{ct}}\rangle_{d=4} = 2W_{\mathrm{CS},d=3} + P_{\mathrm{ct},d=2}, \qquad (162)$$

from which (159) follows.

As we have said, the chirality breaking terms coincide at both sides of (159). On the other hand, equating the chirality preserving terms at both sides gives a relation between the functions N in d dimensions and the functions A in $d+2$ dimensions. ($\mathsf{J}^-_{v,\mathrm{c},d+2}$ is a $d+1$-form, so $W_d^-[v,m]$ must first be brought to a $d+1$-dimensional form by applying $\hat{\mathsf{D}}$.) For $d=0$ and $d=2$ the relations are

$$0 = f_1 = f_{12} - f_{2\underline{1}} = f_{123} - f_{23\underline{1}} + f_{3\underline{12}}, \qquad (163)$$

where

$$f_1 = \frac{1}{\mathsf{m}_1} + \frac{1}{2}A_{1\underline{1}},$$
$$f_{12} = \frac{1}{\mathsf{m}_1} + \frac{1}{\mathsf{m}_2} - 2(\mathsf{m}_1-\mathsf{m}_2)N_{12} - \frac{1}{6}(A_{12\underline{1}} + A_{21\underline{2}}),$$
$$f_{123} = -\frac{1}{3}\frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3} + (\Delta N)_{123} - \frac{1}{6}A_{123\underline{1}}. \qquad (164)$$

These relations are checked by our calculation.

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Appendix A: Chiral anomaly and WZW action

Here we will collect some formulas which are needed in the text. The variation of the effective action under infinitesimal chiral rotations is the (consistent) chiral anomaly. As is well known, W^+ can be renormalized so that it is free from chiral anomalies and hence only the pseudo-parity odd component of the effective action is necessarily anomalous.

Let $\Omega_{\mathrm{L,R}} = \exp(\alpha_{\mathrm{R,L}})$. Then, the LR version of the anomaly takes the form (where α is infinitesimal)

$$\delta W^-_{\mathrm{LR},d=0}[v,m] = -\langle \alpha_{\mathrm{R}} - \alpha_{\mathrm{L}}\rangle = \langle -2\alpha\rangle\,,$$
$$\delta W^-_{\mathrm{LR},d=2}[v,m] = \langle v_{\mathrm{R}}\mathrm{d}\alpha_{\mathrm{R}} - v_{\mathrm{L}}\mathrm{d}\alpha_{\mathrm{L}}\rangle = \langle 2\mathsf{v}\mathrm{d}\alpha\rangle\,, \qquad \text{(A1)}$$
$$\delta W^-_{\mathrm{LR},d=4}[v,m] = \left\langle \left(-4\mathsf{Fv} + 2\mathsf{v}^3\right)\mathrm{d}\alpha\right\rangle.$$

The LR anomaly presents two key features. First, it does not depend on m, and second, the two chiral sectors do not mix. In addition, it is consistent, i.e. a true variation. Let (v,m) be a field configuration obtained from another configuration $(\bar v,\bar m)$ through a chiral rotation $(\Omega_{\mathrm{L}},\Omega_{\mathrm{R}})$, i.e. $(v,m) = (\bar v,\bar m)^\Omega$. Then integration of the anomaly yields

$$W^-_{\mathrm{LR}}[v,m] = W^-_{\mathrm{LR}}[\bar v,\bar m] + \Gamma[v_{\mathrm{R}},\Omega_{\mathrm{R}}] - \Gamma[v_{\mathrm{L}},\Omega_{\mathrm{L}}]. \qquad \text{(A2)}$$

($\Gamma[v,\Omega]$ is the same function in both cases, but with different arguments.) Reflecting the same property of the LR anomaly, the variation is composed of two terms which are not mixed and are independent of m. Explicitly,

$$\Gamma_{d=2}[v,\Omega] = \left\langle -\frac{1}{3}(r_c^3 + v^3) + (r_c+v)F\right\rangle$$
$$= \left\langle -\frac{1}{3}r^3 + vr\right\rangle, \qquad \text{(A3)}$$
$$\Gamma_{d=4}[v,\Omega] = \left\langle -\frac{1}{5}(r_c^5+v^5) + (r_c^3+v^3)F - 2(r_c+v)F^2\right\rangle$$
$$= \left\langle -\frac{1}{5}r^5 + vr^3 + v^2r^2 - \frac{1}{2}(vr)^2v^3r - 2Fvr\right\rangle, \qquad \text{(A4)}$$

where $r = \Omega^{-1}\mathrm{d}\Omega$ and $r_c = \Omega^{-1}\mathrm{d}\Omega - v$.

The VA version of the effective action is characterized by being vector gauge invariant. It is obtained from the LR version by subtracting an appropriate local polynomial counterterm:

$$W^-_{\mathrm{VA}}[v,m] = W^-_{\mathrm{LR}}[v,m] - P_{\mathrm{ct}}[v]. \qquad \text{(A5)}$$

Note that the counterterm is independent of m. Explicitly (Convention 1 applies),

$$P_{\mathrm{ct},d=2}[v] = \langle v_{\mathrm{R}}v_{\mathrm{L}}\rangle\,, \qquad \text{(A6)}$$
$$P_{\mathrm{ct},d=4}[v] = \left\langle 2F_{\mathrm{R}}[v_{\mathrm{L}},v_{\mathrm{R}}] + 2v_{\mathrm{R}}v_{\mathrm{L}}^3 - \frac{1}{2}(v_{\mathrm{R}}v_{\mathrm{L}})^2\right\rangle.$$

The corresponding VA anomaly is thus

$$\delta W^-_{\mathrm{VA},d=2}[v,m] = \left\langle 4\left(F_V - A^2\right)\alpha_A\right\rangle\,, \qquad \text{(A7)}$$
$$\delta W^-_{\mathrm{VA},d=4}[v,m] = \big\langle -4\left(3F_V^2 + F_A^2 - 4AF_VA - \{F_V,A^2\} - A^4\right)\alpha_A\big\rangle\,,$$

where $v_{\mathrm{R,L}} = V \pm A$, $F_V = D_V^2 = \mathrm{d}V + V^2$ and $F_A = \{D, A\}$. In addition, we have introduced the vector and axial variations through $\alpha_{\mathrm{R,L}} = \alpha_V \pm \alpha_A$. As advertised, in this case there is no anomaly associated to vector transformations.

Consider now the variation of the VA effective action

$$W^-_{\mathrm{VA}}[v,m] = W^-_{\mathrm{VA}}[\bar{v},\bar{m}] + \Gamma_{\mathrm{VA}}[v,U], \quad U := \Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{R}}. \tag{A8}$$

$\Gamma_{\mathrm{VA}}[v,U]$ is the gauged WZW action which, by construction, saturates the VA anomaly. Because the anomaly is independent of m, so is $\Gamma_{\mathrm{VA}}[v,U]$. In addition, since W^-_{VA} is vector gauge invariant its variation depends on $\Omega_{\mathrm{L,R}}$ only through the combination $U = \Omega_{\mathrm{L}}^{-1}\Omega_{\mathrm{R}}$, i.e., the axial part of Ω. The LR form of this relation is obtained by adding the counterterm $P_{\mathrm{ct}}[v]$. This gives

$$W^-_{\mathrm{LR}}[v,m] = W^-_{\mathrm{VA}}[\bar{v},\bar{m}] + \Gamma_{\mathrm{LR}}[v,U]. \tag{A9}$$

Note that by construction $\Gamma_{\mathrm{VA}}[v,1] = 0$ and $\Gamma_{\mathrm{LR}}[v,1] = P_{\mathrm{ct}}[v]$, so

$$\Gamma_{\mathrm{VA}}[v,U] = \Gamma_{\mathrm{LR}}[v,U] - \Gamma_{\mathrm{LR}}[v,1]. \tag{A10}$$

$\Gamma_{\mathrm{LR}}[v,1]$ is known as the Bardeen subtraction.

Comparing (A2), (A5) and (A9), it follows that

$$\Gamma_{\mathrm{LR}}[v,U] = \Gamma(v_{\mathrm{R}},\Omega_{\mathrm{R}}) - \Gamma(v_{\mathrm{L}},\Omega_{\mathrm{L}}) + P_{\mathrm{ct}}[\bar{v}]. \tag{A11}$$

On the other hand, noting that the Bardeen subtraction vanishes for purely right or left gauge fields, we find

$$\Gamma[v,\Omega] = \Gamma_{\mathrm{LR}}[v_{\mathrm{R}} = v, v_{\mathrm{L}} = 0, U = \Omega]. \tag{A12}$$

Explicitly, in two dimensions

$$\Gamma_{\mathrm{LR},d=2}[v,U] = \Big\langle -\frac{1}{3}(U^{-1}\mathrm{d}U)^3 - U^{-1}\mathrm{d}U v_{\mathrm{R}} + U\mathrm{d}U^{-1}v_{\mathrm{L}} - U^{-1}v_{\mathrm{L}}Uv_{\mathrm{R}}\Big\rangle. \tag{A13}$$

In order to use the Conventions 1 and 2, let us define U as $(\mathsf{U})_{\mathrm{LR}} = U$ and $(\mathsf{U})_{\mathrm{RL}} = U^{-1}$. Note that U^{-1} equals U with our conventions. In addition, let $\mathsf{R} = \mathsf{U}\mathrm{d}\mathsf{U}$. Then

$$\Gamma_{\mathrm{LR},d=2}[v,U] = \Big\langle -\frac{1}{3}\mathsf{R}^3 - 2\mathsf{R}\mathsf{v} - \mathsf{U}\mathsf{v}\mathsf{U}\mathsf{v}\Big\rangle. \tag{A14}$$

In four dimensions

$$\Gamma_{\mathrm{LR},d=4}[v,U] = \Big\langle -\frac{1}{5}\mathsf{R}^5 - 2\mathsf{R}^3\mathsf{v} + (\mathsf{R}\mathsf{v})^2 + 2\mathsf{R}^2\mathsf{v}\mathsf{U}\mathsf{v}\mathsf{U} + 2\mathsf{R}\mathsf{U}\mathsf{v}\mathsf{U}\mathrm{d}\mathsf{v} + 2\mathsf{R}\mathsf{v}^3 + 2\mathsf{R}\mathsf{v}\mathsf{U}\mathsf{v}\mathsf{U}\mathsf{v} + 2(\mathsf{R}+\mathsf{U}\mathsf{v}\mathsf{U})\{\mathsf{v},\mathrm{d}\mathsf{v}\} + 2\mathsf{U}\mathsf{v}\mathsf{U}\mathsf{v}^3 + \frac{1}{2}(\mathsf{U}\mathsf{v}\mathsf{U}\mathsf{v})^2\Big\rangle. \tag{A15}$$

Appendix B: Explicit formulas for the functions A and N in two and four dimensions

For the currents we give the formulas for the associated functions $\bar{A}$ which are more symmetric. In two dimensions

$$\bar{A}_{12} = -\frac{2}{\mathsf{m}_1+\mathsf{m}_2} - \frac{2\mathsf{m}_1\mathsf{m}_2}{(\mathsf{m}_1+\mathsf{m}_2)(\mathsf{m}_1^2-\mathsf{m}_2^2)}\log(\mathsf{m}_1^2/\mathsf{m}_2^2). \tag{B1}$$

In four dimensions,

$$\bar{A}_{123} = \bar{A}^{\mathrm{R}}_{123} + \bar{A}^{\mathrm{L}}_{123}\log(\mathsf{m}_1^2/\mathsf{m}_3^2) + \bar{A}^{\mathrm{L}}_{213}\log(\mathsf{m}_2^2/\mathsf{m}_3^2) \tag{B2}$$

$$\bar{A}_{1234} = \bar{A}^{\mathrm{R}}_{1234} + \bar{A}^{\mathrm{L}}_{1234}\log(\mathsf{m}_1^2) + \bar{A}^{\mathrm{L}}_{2341}\log(\mathsf{m}_2^2) + \bar{A}^{\mathrm{L}}_{3412}\log(\mathsf{m}_3^2) + \bar{A}^{\mathrm{L}}_{4123}\log(\mathsf{m}_4^2) \tag{B3}$$

(where the superindices R and L refer to rational and logarithmic components, respectively). With

$$\bar{A}^{\mathrm{R}}_{123} = \frac{6(\mathsf{m}_1\mathsf{m}_2+\mathsf{m}_1\mathsf{m}_3+\mathsf{m}_2\mathsf{m}_3)}{(\mathsf{m}_1+\mathsf{m}_2)(\mathsf{m}_1+\mathsf{m}_3)(\mathsf{m}_2+\mathsf{m}_3)} \tag{B4}$$

$$\bar{A}^{\mathrm{L}}_{123} = \frac{6\mathsf{m}_1^3(\mathsf{m}_1\mathsf{m}_2+\mathsf{m}_1\mathsf{m}_3+2\mathsf{m}_2\mathsf{m}_3)}{(\mathsf{m}_1+\mathsf{m}_2)(\mathsf{m}_1+\mathsf{m}_3)(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_1^2-\mathsf{m}_3^2)}, \tag{B5}$$

$$\bar{A}^{\mathrm{R}}_{1234} = \{(6(\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3+\mathsf{m}_1\mathsf{m}_2\mathsf{m}_4+\mathsf{m}_1\mathsf{m}_3\mathsf{m}_4+\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4))/((\mathsf{m}_1+\mathsf{m}_2)(\mathsf{m}_1+\mathsf{m}_3)(\mathsf{m}_1+\mathsf{m}_4)\times(\mathsf{m}_2+\mathsf{m}_3)(\mathsf{m}_2+\mathsf{m}_4)(\mathsf{m}_3+\mathsf{m}_4))\} \tag{B6}$$

$$\bar{A}^{\mathrm{L}}_{1234} = -\{(6\mathsf{m}_1^3(\mathsf{m}_1(\mathsf{m}_2\mathsf{m}_3+\mathsf{m}_2\mathsf{m}_4+\mathsf{m}_3\mathsf{m}_4)+2\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4-\mathsf{m}_1^3))/((\mathsf{m}_1+\mathsf{m}_2)(\mathsf{m}_1+\mathsf{m}_3)\times(\mathsf{m}_1+\mathsf{m}_4)(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_1^2-\mathsf{m}_3^2)\times(\mathsf{m}_1^2-\mathsf{m}_4^2))\}. \tag{B7}$$

For the effective action in two dimensions

$$N_{12} = -\frac{\mathsf{m}_1\mathsf{m}_2}{\mathsf{m}_1^2-\mathsf{m}_2^2}\left(\frac{\log(\mathsf{m}_1^2/\mathsf{m}_2^2)}{\mathsf{m}_1^2-\mathsf{m}_2^2} - \frac{1}{2}\left(\frac{1}{\mathsf{m}_1^2}+\frac{1}{\mathsf{m}_2^2}\right)\right). \tag{B8}$$

In four dimensions, the function N_{123} can be written as

$$N_{123} = N^{\mathrm{R}}_{123} + N^{\mathrm{L}}_{123}\log(\mathsf{m}_1^2/\mathsf{m}_2^2) - N^{\mathrm{L}}_{321}\log(\mathsf{m}_3^2/\mathsf{m}_2^2), \tag{B9}$$

with

$$N^{\mathrm{R}}_{123} = \frac{1}{2\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_3^2-\mathsf{m}_2^2)(\mathsf{m}_1-\mathsf{m}_3)} \times\Big(3\mathsf{m}_1^2\mathsf{m}_3^2(\mathsf{m}_1-\mathsf{m}_3)^2 + 4\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3(\mathsf{m}_1+\mathsf{m}_3)(2\mathsf{m}_1^2-3\mathsf{m}_1\mathsf{m}_3+2\mathsf{m}_3^2-\mathsf{m}_2^2) + \mathsf{m}_2^2(\mathsf{m}_1^4+10\mathsf{m}_1^3\mathsf{m}_3-18\mathsf{m}_1^2\mathsf{m}_3^2+10\mathsf{m}_1\mathsf{m}_3^3+\mathsf{m}_3^4) - \mathsf{m}_2^4(\mathsf{m}_1+\mathsf{m}_3)^2\Big), \tag{B10}$$

$$N^{\mathrm{L}}_{123} = \frac{2}{(\mathsf{m}_1^2-\mathsf{m}_2^2)^2(\mathsf{m}_1^2-\mathsf{m}_3^2)(\mathsf{m}_1-\mathsf{m}_3)} \times\Big(\mathsf{m}_1^4(\mathsf{m}_2-2\mathsf{m}_3) + \mathsf{m}_1^2(\mathsf{m}_2^3+\mathsf{m}_3^3) + \mathsf{m}_2^2\mathsf{m}_3^2(\mathsf{m}_2+\mathsf{m}_3) + \mathsf{m}_1^3(\mathsf{m}_2^2-3\mathsf{m}_2\mathsf{m}_3-\mathsf{m}_3^2) - \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3(\mathsf{m}_2^2-\mathsf{m}_3^2)\Big). \tag{B11}$$

Likewise,

$$N_{1234} = N^{\rm R}_{1234} + N^{\rm L}_{1234}\log(\mathsf{m}_1^2) + N^{\rm L}_{234\underline{1}}\log(\mathsf{m}_2^2) + N^{\rm L}_{34\underline{12}}\log(\mathsf{m}_3^2) + N^{\rm L}_{4\underline{123}}\log(\mathsf{m}_4^2), \qquad \text{(B12)}$$

where

$$\begin{aligned} N^{\rm R}_{1234} = \frac{1}{4}\Bigg(& \frac{2(2\mathsf{m}_2+\mathsf{m}_3)}{(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_2^2-\mathsf{m}_3^2)(\mathsf{m}_2-\mathsf{m}_4)} \\ & - \frac{2(2\mathsf{m}_2+\mathsf{m}_1)}{(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_2^2-\mathsf{m}_3^2)(\mathsf{m}_2+\mathsf{m}_4)} \\ & - \frac{3(\mathsf{m}_2\mathsf{m}_3-\mathsf{m}_1(\mathsf{m}_2+\mathsf{m}_3))}{\mathsf{m}_3(\mathsf{m}_1^2-\mathsf{m}_3^2)(\mathsf{m}_2^2-\mathsf{m}_3^2)(\mathsf{m}_3-\mathsf{m}_4)} \\ & + \frac{3(\mathsf{m}_1\mathsf{m}_2-\mathsf{m}_3(\mathsf{m}_1+\mathsf{m}_2))}{\mathsf{m}_1(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_1^2-\mathsf{m}_3^2)(\mathsf{m}_1+\mathsf{m}_4)} \\ & - \frac{\mathsf{m}_2\mathsf{m}_3+\mathsf{m}_1(\mathsf{m}_2+\mathsf{m}_3)}{\mathsf{m}_1(\mathsf{m}_1^2-\mathsf{m}_2^2)(\mathsf{m}_1^2-\mathsf{m}_3^2)(\mathsf{m}_1-\mathsf{m}_4)} \\ & + \frac{\mathsf{m}_2\mathsf{m}_3+\mathsf{m}_1(\mathsf{m}_2+\mathsf{m}_3)}{\mathsf{m}_3(\mathsf{m}_1^2-\mathsf{m}_3^2)(\mathsf{m}_2^2-\mathsf{m}_3^2)(\mathsf{m}_3+\mathsf{m}_4)} \\ & + \frac{1}{\mathsf{m}_1\mathsf{m}_2\mathsf{m}_3\mathsf{m}_4}\Bigg), \end{aligned} \qquad \text{(B13)}$$

$$\begin{aligned} N^{\rm L}_{1234} = & \frac{1}{2(\mathsf{m}_1^2-\mathsf{m}_2^2)^2(\mathsf{m}_1^2-\mathsf{m}_3^2)^2(\mathsf{m}_1^2-\mathsf{m}_4^2)^2} \\ & \times (6\mathsf{m}_1^7\mathsf{m}_3 + (\mathsf{m}_2-\mathsf{m}_4)(\mathsf{m}_2^2\mathsf{m}_3^3\mathsf{m}_4^2 + 3\mathsf{m}_1^6\mathsf{m}_3) \\ & - \mathsf{m}_1\mathsf{m}_2\mathsf{m}_3^3\mathsf{m}_4(\mathsf{m}_2-\mathsf{m}_4)^2 \\ & + \mathsf{m}_1^2\mathsf{m}_3^2(\mathsf{m}_2^3(2\mathsf{m}_4+\mathsf{m}_3) - \mathsf{m}_4^3(2\mathsf{m}_2+\mathsf{m}_3)) \\ & - \mathsf{m}_1^4(\mathsf{m}_2-\mathsf{m}_4)(2\mathsf{m}_2^2(\mathsf{m}_3+\mathsf{m}_4) \\ & + \mathsf{m}_2\mathsf{m}_4(\mathsf{m}_3+2\mathsf{m}_4) + 2\mathsf{m}_3(\mathsf{m}_3^2+\mathsf{m}_4^2)) \\ & + \mathsf{m}_1^3(-\mathsf{m}_2^3\mathsf{m}_4^3 + \mathsf{m}_2^2\mathsf{m}_4^2(2\mathsf{m}_3+\mathsf{m}_4) \\ & + \mathsf{m}_2\mathsf{m}_3\mathsf{m}_4(2\mathsf{m}_3^2+\mathsf{m}_4^2) + \mathsf{m}_2^3(-\mathsf{m}_3^2+\mathsf{m}_3\mathsf{m}_4+\mathsf{m}_4^2)) \\ & - \mathsf{m}_1^5(\mathsf{m}_2^2(4\mathsf{m}_3+\mathsf{m}_4) + \mathsf{m}_2(-\mathsf{m}_3^2+2\mathsf{m}_3\mathsf{m}_4+\mathsf{m}_4^2) \\ & + \mathsf{m}_3(2\mathsf{m}_3^2-\mathsf{m}_3\mathsf{m}_4+4\mathsf{m}_4^2))). \end{aligned} \qquad \text{(B14)}$$

We have tried to write these formulas in a form as simple as possible. In the case of $N^{\rm R}_{1234}$, this implies that the cyclic property is not manifest.

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